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Canonical filtrations, Hasse invariants and overconvergent modular forms on Shimura varieties with ramified datum

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Introduction

The theory of p-adic modular forms has been introduced to study congruences between modular forms. It has been developed by many authors, especially Serre, which studied modular forms through their q-expansion in [Se]. He defines then p-adic modular forms as p-adic limits of classical modular forms. A key observation is that a p-adic modular form has a weight, which is a p-adic integer.
A geometric interpretation has been given by Katz ([Kat1]), using the modular curve, and the definition of modular forms as sections of modular sheaves on this space. He defines a p-adic modular form as a section of a certain sheaf on a special locus, the ordinary locus. This locus is defined as the locus where the universal elliptic curve is ordinary at p (its p-torsion is an extension of a multiplicative part and an étale part). This is also the non vanishing locus of a certain function, the Hasse invariant (which is the Eisenstein series E_{p-1} in this case).
A deeper notion is the one of overconvergent modular forms, introduced by Katz ([Kat1]) : these are defined as sections on a strict neighborhood of the ordinary locus. They are easily defined for classical weights, as one has a modular sheaf on the whole variety. Defining the sheaves for p-adic weights is a more difficult task, done in [Pi1].
To achieve this, one needs two important tools : the Hasse invariant, and the canonical subgroup. Indeed, one can prove that on a strict neighborhood of the ordinary locus there exists a special subgroup inside the p-torsion of the elliptic curve, called the canonical subgroup. It is of multiplicative type on the ordinary locus, and can be explicitly described on a strict neighborhood.
An important result, proved by Coleman [Col], is that an overconvergent modular form of classical weight is indeed classical if it is an eigenform for a Hecke operator, and the valuation of the eigevalue (the slope) is small enough. Such a result is known as a classicality theorem. Another method has then been given by Buzzard and Kassaei ([Bu], [Ka]), using an analytic continuation method. Starting with an overconvergent modular form with the desired hypothesis, one extends it on the whole variety, and thus proves that it is classical.

This theory has then been developed for more general varieties, especially Hilbert modular forms. One still has an ordinary locus, and one can then define overconvergent modular forms. The definition of the modular sheaves has been done in [AIP], and classicality results have been obtained in [Sa1], [PS], [Bi1]. Actually, one can consider general Shimura varieties ; the construction of the p-adic modular sheaves has been done in [AIP2], [Br] and the classicality result has been obtained in [BPS], [Bi2]. One issue is that in some cases, the ordinary locus may be empty. This situation has been studied in [We] : this locus is non empty if and only if the prime p totally splits in the reflex field (assuming the datum is unramified at p). For Hilbert or Hilbert-Siegel varieties, the reflex field is equal to $\mathbb{Q}$, and this condition is automatically satisfied. However, for other Shimura varieties this may not be the case, and this seems to be an issue to adapt the theory of overconvergent modular forms.

Let us say a few words on the two main tools used in this theory : the Hasse invariant and the canonical subgroup. The Hasse invariant is defined thanks to the Verschiebung, acting on the sheaf of differentials. When one considers a variety with extra structure, one can moreover decompose this function as a product of primitive Hasse invariants. These are quite natural in the unramified setting, and more involved when one allows ramification. The latter situation has been studied in [RX] in the Hilbert case.

Many authors have worked on the canonical subgroup, see for example [AM], [Con], [Fa2]. The result from Fargues states that if the valuation of the Hasse invariant of a p-divisible group is small enough, there exists a special subgroup called the canonical subgroup. He also gives information about this subgroup, especially a relation between its degree and the Hasse invariant.
A drawback of this result is that it is only valid for $p \neq 2$; another one is that it does not give a definition for a canonical subgroup. We give here a different approach : we give a simple definition for the canonical subgroup, prove that if it exists it is unique, and then prove the desired properties. Moreover, when one considers a p-divisible group with an action, one can define partial degrees for any subgroup. We are then able to relate the partial degrees of the canonical subgroup to the primitive Hasse invariants.
An important notion for p-divisible groups is the notion of duality : one can define the Cartier dual of a p-divisible group G. If G has height h and dimension d, then the dual G^D has height h and dimension $h-d$. Fargues explored in [Fa2] the behavior of the canonical subgroup under duality. More precisely, he proved that the Hasse invariants for G and G^D can be identified, and the canonical subgroup for G induces a canonical subgroup for G^D.
We give a simpler proof for this result, and extends it to primitive Hasse invariants. The key point is to introduce the Hodge filtration and the conjugate filtration as in [EV] ; the Hasse invariant is then obtained by the relative position of these subsheaves inside a certain sheaf.

All of this was possible under the assumption that the ordinary locus was non empty. This includes in particular the cases of Hilbert modular varieties and Hilbert-Siegel modular varieties. However, for unitary Shimura varieties, the ordinary locus may be empty. This is in particular the case for the Picard variety with a prime p inert in the quadratic field. Looking at the geometry of the variety, on still finds that one has a special locus, called the μ-ordinary locus. It is an open and dense subset of the special fiber, and is equal to the ordinary locus when this latter is non empty. It has been studied in the case when the prime p is unramified in [We] and [Mo]. One can then try to adapt the theory of overconvergent modular to this context, using the μ-ordinary locus.
In [Bi3], we consider such a variety (in the unramified situation), and show how to generalize our previous approach to the canonical subgroup. The main difference is that one has a canonical filtration in this setting (i.e. several subgroups in the p-torsion, each of them canonical). We give a definition for this notion, and prove that if such a filtration exists, it is necessarily unique. We then give a definition for overconvergent modular forms (of classical weights), and prove a classicality result in this context.
Other works have been done in this direction. First of all, a μ-ordinary Hasse invariant has been constructed in [GN], [KW] and [He]. In [He2], Hernandez proved that if the valuation of the μ-ordinary Hasse invariant is small enough, a canonical filtration exists. Note that this result is only valid for some prime p bigger than a constant. He also gives information about the subgroups of this filtration ; more precisely, he relates a linear combination of their partial degrees to a partial Hasse invariant. Finally, in [He3], Hernandez constructs the modular sheaves of p-adic weights, thus giving a definition for overconvergent modular forms of p-adic weights.
The relation obtained by Hernandez about the subgroups in the canonical filtration is not optimal. Indeed, he can only compute a linear combination of their partial degrees, whereas in the ordinary setting, one can compute all the partial degrees of the canonical subgroup. We solve this question by introducing refined partial Hasse invariants. They are not defined globally, unless one introduces a flag variety. For a p-divisible group over a valuation ring, this gives new invariants, and one can prove that the μ-ordinary Hasse invariant is equal to a product of these refined partial

Hasse invariants. Finally, we relate the partial degrees of each subgroup of the canonical filtration (when it exists) to these refined partial Hasse invariants. Note that this result in unconditional on p.

The previous results were obtained in the case where the prime p is unramified in the datum. The geometry of the Shimura variety has been extensively studied, and many results have been obtained in this case. The case where the prime p ramifies is much deeper. The definition of the variety is itself an issue. Even for the Hilbert modular variety, one can define the variety as in [Ra], with the so called Rapoport condition. The associated space is smooth, but the compactified space is not proper. One can also consider the Deligne-Pappas model ([D-P], which has a proper compactification, but is not smooth. Actually, the smooth locus is exactly the Rapoport locus (i.e. points satisfying the Rapoport condition).
To solve these issues, one can consider the models constructed by Pappas and Rapoport ([PR1], [PR2], [PR3]). Note that the definition of the integral models depends on a choice of an ordering for the embeddings of the totally real field, but the special fiber does not. This definition is actually valid for more general Shimura varieties, especially the ones associated to unitary groups. However, none of the objects accessible in the unramified case (μ-ordinary locus, Hasse invariant, canonical filtration) are accessible, and very little of the geometry of the variety is known.
In [BH1], we consider a p-divisible group with a ramified action over a perfect field of characteristic p, with a PR datum, thus making a local study of the situation encountered in the above varieties. We define the Newton and Hodge polygons, and prove that they lie above a certain polygon depending on the datum, that we call the PR polygon. We say that the group is μ-ordinary if the Newton polygon is equal to the PR polygon ; it states that the structure of the p-divisible group is the best possible given the imposed conditions. If the Hodge polygon is equal to the PR polygon, we say that the group satisfies the generalized Rapoport condition. This means that the structure of the Lie algebra is the best possible given the imposed conditions. This notion coincides with the Rapoport condition in the Hilbert case, hence the terminology.
We then study in further details the μ-ordinariness notion : we prove that over an algebraically closed field, it is isomorphic to an explicit group. We also construct a μ-ordinary Hasse invariant, i.e. a section of a sheaf such that the group is μ-ordinary if and only if this section is non zero.
Once this local study achieved, we turned to the global point of view, and to the geometry of the Shimura varieties defined by the Pappas-Rapoport condition. In particular, we prove in [BH2] that the varieties are smooth. We also prove that the μ-ordinary locus, and the generalized Rapoport locus are dense.
One big difference with the unramified case is that the Hodge polygon we define varies on the variety. One idea is thus to try to define a stratification on this variety using the Hodge polygon. The problem is that the naive approach only gives a weak stratification (the closure of a stratum is included but may not be equal to a union of strata). One would then need to define a finer invariant that the Hodge polygon. One could think about the isomorphism class of the Lie algebra with the PR datum. However, this does not give a finite number of strata.
We solve this question in [Bi7] for ramification index e less or equal than 3. The problem is empty when it is equal to 1, and the Hodge polygon induces a strong stratification when it is equal to 2. When $e = 3$, we show that one should consider the total Hodge polygon, as well as the Hodge polygon of the graded parts. The stratification is thus indexed by three polygons, and gives indeed a strong stratification.
We present here some conjectures about the canonical filtration in the ramified case. One big difference with the unramified setting is that when one considers a p-divisible group G with a ramified

action, one can consider the π-torsion, instead of the p-torsion. One possible definition of the canonical filtration would be to define a filtration on the π-torsion. However, the subgroups introduced are not related to the Hasse invariants constructed in this context. Our solution is to consider a filtration $0 \subseteq C_1 \cdots \subseteq C_e$ such that C_k belongs to the π^k-torsion of G. We then express some conjectural relations between the degrees of these subgroups and the Hasse invariants.

Finally, we turn to the study of overconvergent modular forms for Shimura varieties with ramified datum. We use our definition of the μ-ordinary locus, and the canonical filtration to give a definition for overconvergent modular forms of classical weights. We introduce what we believe to be the relevant Hecke operator acting on both classical and overconvergent modular forms. We then express a classicality conjecture.

The interest of overconvergent modular forms is of course to be able to consider families of such forms, and to consider overconvergent modular forms of non classical weight. We hope that this could be achieved, generalizing the works of [AIP2], [Br], and then to construct eigenvarieties in this setting.
Let us mention two topics of research developed recently, which would be interesting to further study. First, there is the theory of adic eigenvarieties, developed by Andreatta, Iovita and Pilloni in [AIP3] and [AIP4]. These are used to consider overconvergent modular forms in characterstic p. They need to use adic spaces instead of a rigid-analytic variety.
Secondly, we mention the development of higher Hida theory ([BP], [Gr]), when one should consider the spaces of cohomology $H^i(X, \omega^\kappa)$, with $i > 0$. When $i = 0$, this is the theory of overconvergent modular forms, and the authors aim to develop a similar theory in all cohomological degree.

Let us now present the organization of the text. In section 1, we give the definition and properties of a p-divisible group with a ramified action and a PR datum over a perfect field of characteristic p. In particular, we define the notion of μ-ordinariness. In section 2, we explain the construction of the μ-ordinary Hasse invariants, and study the relation with duality. Section 3 is devoted to the theory of the canonical subgroup and the canonical filtration. We explain the definition and properties in the unramified case, and give some conjectures in the general case. We turn to the global study in section 4. We consider varieties defined thanks to the PR datum, and study their geometry. Finally, we give some applications to overconvergent modular forms in section 5. In the context of a ramified action, the statements are still conjectures.

We fix once and for all a prime p.

1 p-divisible groups with Pappas-Rapoport datum

1.1 Hodge and Newton polygons

In this section, we introduce some notations, and the main objects that we study.

Let k be a perfect field of characteristic p, and let $W(k)$ be the ring of Wit vectors of k. Let σ be the Frobenius acting on $W(k)$. Let G be a p-divisible group over k. To G one can attach its Dieudonné module (M, F, V) : M is a free $W(k)$-module of finite rank, $F : M \to M$ is a σ-linear map, $V : M \to N$ is a σ^{-1}-linear map such that $FV = VF = p\,\mathrm{id}$ (see [Fo] part III).

Let L be a finite extension of $\mathbb{Q}_p$, L^{ur} the maximal unramified extension contained in L and k_L the residue field of L. Let f be the residual degree, e the ramification index and π a uniformizer in L. Let O_L and $O_{L^{ur}} = W(k_L)$ be the ring of integers of L and L^{ur}, and we suppose that k contains k_L.

Definition 1.1.1. *An action of O_L on G is a morphism*

$$O_L \to End(G).$$

We suppose that the p-divisible group G has an action of O_L. This induces an action of O_L on the Dieudonné module M. In particular, M has an action of $O_{L^{ur}}$; if $\mathcal{T}$ denotes the set of embeddings from $O_{L^{ur}}$ to $W(k)$, one has a natural decomposition

$$M = \bigoplus_{\tau \in \mathcal{T}} M_\tau$$

where M_τ is the submodule of M where $O_{L^{ur}}$ acts by τ. The Frobenius F induces σ-linear maps

$$F_\tau : M_{\sigma^{-1}\tau} \to M_\tau.$$

Similarly, the Verchiebung V induces σ^{-1}-linear maps $V_\tau : M_\tau \to M_{\sigma^{-1}\tau}$. The modules $(M_\tau)_{\tau\in\mathcal{T}}$ are then free $W(k)$-modules with the same rank. Let us define for $\tau \in \mathcal{T}$ the ring of ramified Witt vectors

$$W_{O_L,\tau}(k) := W(k) \otimes_{O_{L^{nr}},\tau} O_L.$$

The ring $W_{O_L,\tau}(k)$ is a discrete valuation ring with uniformizer π. The valuation is normalized on this ring by $v(p) = 1$. The morphism σ extends to a morphism $W_{O_L,\tau}(k) \to W_{O_L,\sigma\tau}(k)$ by $\sigma(\pi) = \pi$.
The modules M_τ are then free over $W_{O_L,\tau}(k)$ of rank h which is independent of τ. We call h the normalized height of G (the usual height being efh).
Let us now define the Hodge and Newton polygons for G. Let $\tau \in \mathcal{T}$; from the elementary divisors theorem, applied to the modules $F_\tau M_{\sigma^{-1}\tau} \subseteq M_\tau$, there exist elements $a_{\tau,1}, \dots, a_{\tau,h}$ in $W_{O_L,\tau}(k)$ such that

$$M_\tau / F_\tau M_{\sigma^{-1}\tau} \simeq \bigoplus_{i=1}^{h} W_{O_L,\tau}(k)/a_{\tau,i} W_{O_L,\tau}(k)$$

One can of course suppose that the valuations of the elements $a_{\tau,i}$ are ordered :
$v(a_{\tau,1}) \leq v(a_{\tau,2}) \leq \dots \leq v(a_{\tau,h})$.

Definition 1.1.2. *The Hodge polygon of G relatively to τ is the polygon on $[0,h]$ whose break points have integral x-coordinates defined by* $\mathrm{Hdg}_{O_L,\tau}(G)(0)=0$ *and*

$$\mathrm{Hdg}_{O_L,\tau}(G)(i)=v(a_{\tau,1})+\cdots+v(a_{\tau,i})$$

for $1\leq i\leq h$. *The Hodge polygon of G is defined as the mean of the polygons* $\mathrm{Hdg}_{O_L,\tau}(G)$, *i.e.*

$$\mathrm{Hdg}_{O_L}(G)(i)=\frac{1}{f}\sum_{\tau\in\mathcal{T}}\mathrm{Hdg}_{O_L,\tau}(G)(i)$$

for $0\leq i\leq h$.

The initial and terminal points of $\mathrm{Hdg}_{O_L,\tau}(G)$ are then $(0,0)$ and $(h,v(\det F_\tau))$, where the determinant of F_τ is seen as en element of $W_{O_L,\tau}(k)$.

Let us consider the morphism $F^f : M_\tau \to M_\tau$. Up to enlarging the field $W_{O_L,\tau}(k)[1/p]$ to a field K by adding roots of π, one can assume that the matrix of F^f acting on M_τ in a certain basis has the form

$$\begin{pmatrix}\lambda_1 & \dots & \star\\ & \ddots & \vdots\\ & & \lambda_h\end{pmatrix}$$

where $\lambda_i\in K$ with $v(\lambda_1)\leq\cdots\leq v(\lambda_h)$.

Definition 1.1.3. *The Newton polygon of G is the polygon on $[0,h]$ whose break points have integral x-coordinates defined by* $\mathrm{Newt}_{O_L}(G)(0)=0$ *and*

$$\mathrm{Newt}_{O_L}(G)(i)=\frac{v(\lambda_1)+\cdots+v(\lambda_i)}{f}$$

for $1\leq i\leq h$.

Dieudonné-Manin theory proves that this polygon is well defined, and is independent of the choice of the embedding τ. The initial and terminal points of this polygon are the same as those of $\mathrm{Hdg}_{O_L}(G)$.

Remark 1.1.4. *If* $\mathrm{Newt}(G)$ *et* $\mathrm{Hdg}(G)$ *are the Newton and Hodge polygons of G (without taking into account the action of O_L), the slopes of* $\mathrm{Newt}(G)$ *are exactly those of* $\mathrm{Newt}_{O_L}(G)$, *each with multiplicity ef. However, the slopes of* $\mathrm{Hdg}(G)$ *and* $\mathrm{Hdg}_{O_L}(G)$ *are not related in general.*

1.2 Pappas-Rapoport datum

We suppose that we are given a collection of integers $\mu=(d_{\tau,i})_{\tau\in\mathcal{T},1\leq i\leq e}$. A Pappas-Rapoport datum for G is defined as follows. Let ω_G be the dual of the Lie algebra of G. We recall that one has an identification $\omega_G\simeq M/FM$, and that one has a decomposition

$$\omega_G=\bigoplus_{\tau\in\mathcal{T}}\omega_{G,\tau}$$

with $O_{L^{ur}}$ acting by τ on $\omega_{G,\tau}$.

Definition 1.2.1. *A Pappas-Rapoport datum for G (PR in short) for the integers μ is a filtration for each $\tau \in \mathcal{T}$*

$$0 = \omega_{G,\tau}^{[0]} \subseteq \omega_{G,\tau}^{[1]} \subseteq \omega_{G,\tau}^{[2]} \subseteq \cdots \subseteq \omega_{G,\tau}^{[e]} = \omega_{G,\tau}$$

such that

- *$\omega_{G,\tau}^{[i]}$ is a vector subspace of $\omega_{G,\tau}$ for all $1 \leq i \leq e$ and $\tau \in \mathcal{T}$.*
- *$\pi \cdot \omega_{G,\tau}^{[i]} \subseteq \omega_{G,\tau}^{[i-1]}$ for all $1 \leq i \leq e$ and $\tau \in \mathcal{T}$.*
- *if we write $\mathrm{Gr}^{[i]}\,\omega_{G,\tau} := \omega_{G,\tau}^{[i]}/\omega_{G,\tau}^{[i-1]}$, then $\dim_k \mathrm{Gr}^{[i]}\,\omega_{G,\tau} = d_{\tau,i}$ for all $1 \leq i \leq e$ and $\tau \in \mathcal{T}$.*

Let $\tau \in \mathcal{T}$; for every $1 \leq i \leq e$, let us consider the convex polygon on $[0, h]$ whose break points have integral x-coordinates with slopes 0 with multiplicity $h - d_{\tau,i}$ and $1/e$ with multiplicity $d_{\tau,i}$. We define the Pappas-Rapoport polygon $\mathrm{PR}_\tau(\mu)$ as the mean of these polygons for $1 \leq i \leq e$. Explicitly, one has

$$\mathrm{PR}_\tau(\mu)(j) = \frac{1}{e}\sum_{i=1}^{e} \max(j - h + d_{\tau,i}, 0).$$

Looking at the length of $M_\tau/F_\tau M_{\sigma^{-1}\tau}$ as $W_{O_L,\tau}(k)$-module, one sees that $\mathrm{Hdg}_{O_L,\tau}(G)(h) = \mathrm{PR}_\tau(\mu)(h)$. In other words, the polygons $\mathrm{Hdg}_{O_L,\tau}(G)$ and $\mathrm{PR}_\tau(\mu)$ have the same initial and terminal points. We define the polygon $\mathrm{PR}(\mu)$ as the mean of the polygons $\mathrm{PR}_\tau(\mu)$ for $\tau \in \mathcal{T}$. The polygons $\mathrm{Hdg}_{O_L}(G)$, $\mathrm{Newt}_{O_L}(G)$ and $\mathrm{PR}(\mu)$ have the same start and end points.

1.3 Properties

If G is a p-divisible group over k with an action of O_L, and with a PR datum for the integers $\mu = (d_\bullet)$, one has three polygons : the Hodge polygon, the Newton polygon and the Pappas-Rapoport polygon. Moreover, there exist relations between these polygons. If P_1 and P_2 are two polygons on $[0, h]$, we say that $P_1 \geq P_2$ if $P_1(x) \geq P_2(x)$ for $x \in [0, h]$.

Theorem 1.3.1 ([BH1] Th. 1.3.1)**.** *Let G be a p-divisible group over k with an action of O_L, and with a PR datum for the integers $\mu = (d_\bullet)$. The one has the inequalities*

$$\mathrm{Newt}_{O_L}(G) \geq \mathrm{Hdg}_{O_L}(G) \geq \mathrm{PR}(\mu).$$

One also has a Hodge-Newton decomposition result in this context. We refer to [Kat2] Th. 1.6.1 for the calssical Hodge-Newton decomposition result.

Theorem 1.3.2 ([BH1] Th. 1.3.2)**.** *Let G be a p-divisible group with action of O_L with normalized height h. Let $a_{\tau,1} \leq \cdots \leq a_{\tau,h}$ be the slopes of $\mathrm{Hdg}_{O_L,\tau}(G)$ for $\tau \in \mathcal{T}$ and $\lambda_1 \leq \cdots \leq \lambda_h$ the ones for $\mathrm{Newt}_{O_L}(G)$. Assume that there exists a point $(i, j) \in \mathbb{N} \times \frac{1}{ef}\mathbb{N}$ which is a break point of $\mathrm{Newt}_{O_L}(M, F)$ and that lies on $\mathrm{Hdg}_{O_L}(G)$. Then there exists a unique decomposition $G = G_1 \times G_2$, where G_1, G_2 are p-divisible groups with action of O_L and such that*

- *G_1 has normalized height i, $\mathrm{Newt}_{O_L}(G_1)$ has slopes $\lambda_1, \ldots, \lambda_i$ and $\mathrm{Hdg}_{O_L,\tau}(G_1)$ has slopes $a_{\tau,1}, \ldots, a_{\tau,i}$ for every $\tau \in \mathcal{T}$.*
- *G_2 has normalized height $h - i$,$\mathrm{Newt}_{O_L}(G_2)$ has slopes $\lambda_{i+1}, \ldots, \lambda_h$ and $\mathrm{Hdg}_{O_L,\tau}(G_2)$ has slopes $a_{\tau,i+1}, \ldots, a_{\tau,h}$ for every $\tau \in \mathcal{T}$.*

Let us introduce the following notions.

Definition 1.3.3. *We say that G satisfies the generalized Rapoport condition if* $\mathrm{Hdg}_{O_L}(G) = \mathrm{PR}(\mu)$.
We say that G is μ-ordinary if $\mathrm{Newt}_{O_L}(G) = \mathrm{PR}(\mu)$.

Remark 1.3.4. *Assume that all the integers $d_{\tau,i}$ are equal ; we refer to this as the ordinary case. Then the polygon $\mathrm{PR}(\mu)$ has slopes 0 and 1 in this case. The generalized Rapoport condition is thus equivalent to the fact that ω_G is free as a $k \otimes_{\mathbb{Z}_p} O_L$-module, which is often referred as the Rapoport condition. The notion of μ-ordinariness is the usual ordinariness notion in this case.*

One has the following characterization of being μ-ordinary.

Theorem 1.3.5 ([BH1] Th. 3.2.1)**.** *Assume that k is algebraically closed. There exists a p-divisible group X^{ord} such that the following assertions are equivalent.*
— *G is μ-ordinary*
— *G is isomorphic to X^{ord}*
— *$G[p]$ is isomorphic to $X^{ord}[p]$.*

Let us say a few words on the p-divisible group X^{ord}. It is expressed as a product $X^{ord} = G_1 \times \cdots \times G_h$, each G_i being of normalized height 1.Each G_i is defined explicitly with its Diedonné module. Let us be more precise.
Let $\beta = (\beta_\tau)_{\tau \in \mathcal{T}}$ be integers with $0 \leq \beta_\tau \leq e$ for each $\tau \in \mathcal{T}$. We define the Dieudonné module (M_β, F, V) by $M_\beta = \bigoplus_{\tau \in \mathcal{T}} M_{\beta,\tau}$ with $M_{\beta,\tau} = W_{O_L,\tau}(k)$ for every $\tau \in \mathcal{T}$. The Frobenius $F_\tau : M_{\beta,\sigma^{-1}\tau} \to M_{\beta,\tau}$ and the Verschiebung $V_\tau : M_{\beta,\tau} \to M_{\beta,\sigma^{-1}\tau}$ are defined by

$$F_\tau(x) = \pi^{\beta_\tau}\sigma(x) \text{ et } V_\tau(y) = p\pi^{-\beta_\tau}\sigma^{-1}(y)$$

for every $\tau \in \mathcal{T}$, $x \in M_{\beta,\sigma^{-1}\tau}$ and $y \in M_{\beta,\tau}$. Remark that the valuation of $p\pi^{-\beta_\tau}$ is equal to $1 - \beta_\tau/e \geq 0$.
This module has an action of O_L, and has normalized rank 1. Let X_β be the p-divisible group associated to the Dieudonné module (M_β, F, V). It is a p-divisible group over k of height ef with an action of O_L. The polygons $\mathrm{Newt}_{O_L}(X_\beta)$ and $\mathrm{Hdg}_{O_L}(X_\beta)$ are equal and have one slope equal to $(\sum_{\tau \in \mathcal{T}} \beta_\tau)/(ef)$.
Let r be the cardinality of the set $\{d_{\tau,i}, \tau \in \mathcal{T}, 1 \leq i \leq e\} \cap [1, h-1]$. We write

$$0 < D_1 < \cdots < D_r < h$$

the elements of this set. Let $D_0 = 0$ and $D_{r+1} = h$. Let $1 \leq j \leq r+1$; for $\tau \in \mathcal{T}$, we define $\alpha_{j,\tau}$ to be the cardinality of the set $\{1 \leq i \leq e, d_{\tau,i} \geq D_j\}$. We define $\alpha_j = (\alpha_{j,\tau})_{\tau \in \mathcal{T}}$; since $\alpha_{j,\tau}$ are between 0 and e, one has the p-divisible group X_{α_j} over k. Remark that $\alpha_{j+1} \leq \alpha_j$ for every $1 \leq j \leq r$.

Definition 1.3.6. *We define the p-divisible X^{ord} over k by*

$$X^{ord} := \prod_{j=1}^{r+1} X_{\alpha_j}^{D_j - D_{j-1}}$$

2 Hasse invariants

2.1 Ordinary case

Let S be a k_L-scheme, and let $G \to S$ be a p-divisible group of dimension d_0 and height h_0. The classical Hasse invariant is obtained thanks to the Verschiebung

$$V : \omega_G \to \omega_G^{(p)}$$

where ω_G is the dual of the Lie algebra of G, and the superscript denotes a twist by the Frobenius. The Hasse invariant of G, $ha(G)$ is obtained by taking the determinant of the above map, and is then a section of the sheaf $\det(\omega_G)^{p-1}$. By convention, this sheaf is the trivial one if $d_0 = 0$, and the Hasse invariant is then equal to 1 in this case.
Let G^D be the Cartier dual of G ; it has dimension $h_0 - d_0$ and height h_0. One has the following compatibility for the Hasse invariant.

Theorem 2.1.1. *We have an isomorphism $\omega_G^{p-1} \simeq \omega_{G^D}^{p-1}$. The elements $ha(G)$ and $ha(G^D)$ are identified under the induced isomorphism $H^0(S, \omega_G^{p-1}) \simeq H^0(S, \omega_{G^D}^{p-1})$.*

This was proved by Fargues in [Fa2] Proposition 2. The simplest way to prove this is to introduce the sheaf $\mathcal{E}$, the evaluation of the contravariant Diedudonné crystal of G at S (see [BBM] section 3.3). It is a locally free sheaf of rank h_0 on S. Moreover, one has the exact sequence

$$0 \to \omega_G \to \mathcal{E} \to \omega_{G^D}^\vee \to 0$$

the last term being the dual of ω_{G^D}. If $\mathcal{F}$ denotes the Hodge filtration (which induces the above exact sequence), the key is to introduce the conjugate filtration $\widetilde{\mathcal{F}} = \operatorname{Ker} V$. The Hasse invariant of G is then obtained by taking the determinant of the natural map $\mathcal{F} \to \mathcal{E}/\widetilde{\mathcal{F}}$, whereas the Hasse invariant of G^D is obtained with the determinant of the natural map $\widetilde{\mathcal{F}} \to \mathcal{E}/\mathcal{F}$ (see [Bi6] Th. 2.1.4).

When one considers p-divisible groups with an action, one can say more : indeed the classical Hasse invariant can be expressed as a product of primitive Hasse invariants. Let us first define the PR datum in this context. Assume that G has an action of O_L. This implies the decomposition

$$\omega_G = \bigoplus_{\tau \in \mathcal{T}} \omega_{G,\tau}$$

with $O_{L^{ur}}$ acting by τ on $\omega_{G,\tau}$. Now let us define the PR datum for G. Let $\mu = (d_{\tau,i})_{\tau \in \mathcal{T}, 1 \leq i \leq e}$ be a collection of integers.

Definition 2.1.2. *A PR datum for G is a filtration $0 = \omega_{G,\tau}^{[0]} \subseteq \omega_{G,\tau}^{[1]} \subseteq \cdots \subseteq \omega_{G,\tau}^{[e-1]} \subseteq \omega_{G,\tau}^{[e]} = \omega_{G,\tau}$ for each $\tau \in \mathcal{T}$, such that*

- *each $\omega_{G,\tau}^{[j]}$ is locally a direct summand of $\omega_{G,\tau}$*
- *$\omega_{G,\tau}^{[j]} / \omega_{G,\tau}^{[j-1]}$ is locally free over $\mathcal{O}_S$ of rank $d_{\tau,j}$ for all $1 \leq j \leq e$.*
- *$\pi \cdot \omega_{G,\tau}^{[j]} \subseteq \omega_{G,\tau}^{[j-1]}$ for all $1 \leq j \leq e$.*

The first remark is that if the integers $d_{\tau,i}$ are not all equal, then the classical Hasse invariant is automatically 0.

Proposition 2.1.3. *Assume that there exists $(\tau, i) \neq (\tau', j)$ such that $d_{\tau,i} \neq d_{\tau',j}$. Then $ha(G) = 0$.*

Démonstration. The Verschiebung sends $\omega_{G,\tau}^{[e]}/\omega_{G,\tau}^{[e-1]}$ to $(\omega_{G,\sigma^{-1}\tau}^{[e]}/\omega_{G,\sigma^{-1}\tau}^{[e-1]})^{(p)}$ and actually this maps factors as

$$\omega_{G,\tau}^{[e]}/\omega_{G,\tau}^{[e-1]} \to \omega_{G,\tau}^{[e-1]}/\omega_{G,\tau}^{[e-2]} \to \cdots \to \omega_{G,\tau}^{[1]} \to (\omega_{G,\sigma^{-1}\tau}^{[e]}/\omega_{G,\sigma^{-1}\tau}^{[e-1]})^{(p)}$$

This facotrization will be detailed later in the section. Therefore, a necessary condition for $ha(G)$ to be non zero is that all these sheaves have the same rank, hence the result. □

Assume in the rest of this section that the integers $d_{\tau,i}$ are all equal to an integer d.
Define $\mathcal{L}_\tau^{[j]} := \det(\omega_{G,\tau}^{[j]}/\omega_{G,\tau}^{[j-1]})$. The classical Hasse invariant then decomposes into primitive Hasse invariants, see [RX].

Proposition 2.1.4. *The classical Hasse invariant can be expressed as a product of the primitive Hasse invariants*

$$m_\tau^{[j]} \in H^0(S, \mathcal{L}_\tau^{[j-1]} {\mathcal{L}_\tau^{[j]}}^{-1}) \qquad hasse_\tau \in H^0(S, (\mathcal{L}_{\sigma^{-1}\tau}^{[e]})^p {\mathcal{L}_\tau^{[1]}}^{-1})$$

for $\tau \in \mathcal{T}$ and $2 \leq j \leq e$.

Let us say a few words about the definition of these primitive Hasse invariants. The multiplication by π induces a map

$$\omega_{G,\tau}^{[j]}/\omega_{G,\tau}^{[j-1]} \to \omega_{G,\tau}^{[j-1]}/\omega_{G,\tau}^{[j-2]}$$

for $\tau \in \mathcal{T}$ and $2 \leq j \leq e$. The section $m_\tau^{[j]}$ is obtained by taking the determinant of the above map. The definition of $hasse_\tau$ is more involved. The sheaf $\mathcal{E}$ decomposes as $\mathcal{E} = \oplus_{\tau \in \mathcal{T}} \mathcal{E}_\tau$, and each $\mathcal{E}_\tau$ is locally free over $\mathcal{O}_S[X]/X^e$, with X acting by π. The multiplication by π^{e-1} thus induces an isomorphism between $\mathcal{E}_\tau/\mathcal{E}_\tau[\pi^{e-1}] \simeq \mathcal{E}_\tau[\pi]$. Let us consider the following map

$$\omega_{G,\tau}^{[1]} \hookrightarrow \mathcal{E}_\tau[\pi] \simeq \mathcal{E}_\tau/\mathcal{E}_\tau[\pi^{e-1}] \to^V (\omega_{G,\sigma-1\tau}^{[e]}/\omega_{G,\sigma^{-1}\tau}^{[e-1]})^{(p)}$$

where the middle isomorphism is the division by π^{e-1} and the last map is the Verschiebung. This map is well defined, and taking its determinant gives the section $hasse_\tau$.

One has the following result concerning the compatibility with duality. Before stating the result, let us remark that a PR datum for G naturally induces a PR datum for G^D. Indeed, one has a full filtration inside $\mathcal{E}_\tau$

$$0 = \omega_{G,\tau}^{[0]} \subseteq \omega_{G,\tau}^{[1]} \subseteq \cdots \subseteq \omega_{G,\tau}^{[e-1]} \subseteq \omega_{G,\tau}^{[e]} = \omega_{G,\tau} \subseteq \omega_{G,\tau}^{[e+1]} \subseteq \cdots \subseteq \omega_{G,\tau}^{[2e]} = \mathcal{E}_\tau$$

where $\omega_{G,\tau}^{[e+i]} := \pi^{-i}\omega_{G,\tau}^{[e-i]}$. Indeed, the multiplication by π^i induces an isomorphism $\mathcal{E}_\tau/\mathcal{E}_\tau[\pi^i] \simeq \mathcal{E}_\tau[\pi^{e-i}]$. Since $\omega_{G,\tau}^{[e-i]}$ is a locally free sheaf inside $\mathcal{E}[\pi^{e-i}]$, the sheaf $\omega_{G,\tau}^{[e+i]}$ is a locally free sheaf. One has then a filtration on $\mathcal{E}_\tau/\omega_{G,\tau} \simeq \omega_{G^D,\tau}^\vee$, and thus on $\omega_{G^D,\tau}$. One easily checks that this is a PR datum for G^D.

Proposition 2.1.5 ([Bi5] Th. 3.7, 3.14)**.** *The sections $m_\tau^{[j]}$ and $hasse_\tau$ are compatible with duality. More precisely, one has isomorphisms of sheaves*

$$\mathcal{L}_{G,\tau}^{[j-1]}{\mathcal{L}_{G,\tau}^{[j]}}^{-1} \simeq \mathcal{L}_{G^D,\tau}^{[j-1]}{\mathcal{L}_{G^D,\tau}^{[j]}}^{-1} \qquad (\mathcal{L}_{G,\sigma^{-1}\tau}^{[e]})^p {\mathcal{L}_{G,\tau}^{[1]}}^{-1} \simeq (\mathcal{L}_{G^D\sigma^{-1}\tau}^{[e]})^p {\mathcal{L}_{G^D,\tau}^{[1]}}^{-1}$$

under which the primitive Hasse invariants for G and G^D are identified.

The result for the sections $hasse_\tau$ is actually proved under some mild assumptions on S (namely, that either S is smooth or locally the spectrum of a semi-perfect ring).

2.2 μ-ordinary Hasse invariants

In this section, we remove the assumption that all the $d_{\tau,i}$ are equal. The classical Hasse invariant may be zero in this case, but one can construct a substitution, the μ-ordinary Hasse invariant. Let us assume that the $d_{\tau,i}$ are ordered :

$$d_{\tau,1} \geq \cdots \geq d_{\tau,e}$$

Proposition 2.2.1 ([BH1] section 2.2)**.** *The exist a section $\mathrm{Ha}_\tau^{[i]} \in H^0(S, {\mathcal{L}_\tau^{[i]}}^{p^f-1})$. The product of these sections give the μ-ordinary Hasse invariant*

$$^{\mu}\,\mathrm{Ha} \in H^0(S, \det(\omega_G)^{p^f-1})$$

Let us say more about the construction of these sections $\mathrm{Ha}_\tau^{[i]}$. The idea is to take the composition of the Verschiebung V^f, and to divide by a suitable power of π. When one considers a p-divisible group over a perfect field k, one can look at its Dieudonné module, which somehow gives a lift of the Verschiebung in characteristic 0, and one can make the division there. Let us be more precise about this situation. Assume that G is defined over k, and let D be the contravariant Diedudonné crystal evaluated at $W(k)$. It is equal to the usual Dieudonné module twisted by the Frobenius (see [BBM] Theorem 4.2.14). This module decomposes as a direct sum $\oplus_{\tau\in\mathcal{T}} D_\tau$, and one has the Hodge filtration $\omega_{G,\tau} \subseteq D_\tau/pD_\tau$. Let $\mathrm{Fil}^{[i]} D_\tau$ be the inverse image of $\omega_{G,\tau}^{[i]}$ for $1 \leq i \leq e$ via the projection $D_\tau \to D_\tau/pD_\tau$.
For any integer d, one can thus construct a map $\zeta_\tau^d : \bigwedge^d D_{\sigma\tau} \to \bigwedge^d D_\tau$, which is equal to $\bigwedge^d V$ divided by $\pi^{k_{\tau,d}}$ with $k_{\tau,d} = \sum_{i=1}^e \max(d - d_{\tau,i}, 0)$. This map is defined using the following diagram.

$$\begin{array}{ccccc}
\bigwedge^d D_{\sigma\tau} & \xrightarrow{\bigwedge^d V_{\sigma\tau}} & \bigwedge^d \mathrm{Fil}^{[e]} D_\tau & & \bigwedge^d D_\tau \\
 & & \downarrow \pi^{\min(d,d_{\tau,e})} & & \\
 & & \bigwedge^d \mathrm{Fil}^{[e-1]} D_\tau & & \\
 & & \downarrow \pi^{\min(d,d_{\tau,2})} \circ \cdots \circ \pi^{\min(d,d_{\tau,e-1})} & \nearrow \frac{1}{\pi^{ed}} & \\
 & & \bigwedge^d \mathrm{Fil}^{[1]} D_\tau & & \\
 & & \downarrow \pi^{\min(d,d_{\tau,1})} & & \\
 & & \bigwedge^d \mathrm{Fil}^{[0]} D_\tau & &
\end{array}$$

One can also construct a map $Div_{\tau,i} : \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} D_\tau \to \bigwedge^{d_{\tau,i}} D_\tau$ as follows

$$\begin{array}{ccc}
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} D_\tau & & \bigwedge^{d_{\tau,i}} D_\tau \\
\downarrow {\scriptstyle \pi^{\min(d_{\tau,i},d_{\tau,i})} = \pi^{d_{\tau,i}}} & & \\
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i-1]} D_\tau & & \\
\vdots {\scriptstyle \pi^{\min(d_{\tau,i},d_{\tau,2})} \circ \cdots \circ \pi^{\min(d_{\tau,i},d_{\tau,i-1})}} & \nearrow {\scriptstyle \frac{1}{\pi^{e d_{\tau,i}}}} & \\
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[1]} D_\tau & & \\
\downarrow {\scriptstyle \pi^{\min(d_{\tau,i},d_{\tau,1})}} & & \\
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[0]} D_\tau & &
\end{array}$$

Finally, let us define a map $Mul_{\tau,i} : \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[e]} D_\tau \to \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} D_\tau$ as the composition

$$\begin{array}{c}
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[e]} D_\tau \\
\downarrow {\scriptstyle \pi^{\min(d_{\tau,i},d_{\tau,e})}} \\
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[e-1]} D_\tau \\
\vdots {\scriptstyle \pi^{\min(d_{\tau,i},d_{\tau,i+2})} \circ \cdots \circ \pi^{\min(d_{\tau,i},d_{\tau,e-1})}} \\
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i+1]} D_\tau \\
\downarrow {\scriptstyle \pi^{\min(d_{\tau,i},d_{\tau,i+1})}} \\
\bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} D_\tau
\end{array}$$

The map $\mathrm{Ha}_\tau^{[i]} : \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} D_\tau \to \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} D_\tau$ is then defined as the composition

$$Mul_{\tau,i} \circ (\bigwedge^{d_{\tau,i}} V_{\sigma\tau}) \circ \zeta_{\sigma\tau}^{d_{\tau,i}} \circ \cdots \circ \zeta_{\sigma^{-1}\tau}^{d_{\tau,i}} \circ Div_{\tau,i}$$

Let us now turn to the general case, and an arbitrary base S. One can consider the sheaf $\mathcal{E} = \oplus \mathcal{E}_\tau$, but it has no lift to characteristic 0. Unfortunately, one cannot divide by π as easily as before, but one can still use the division by π, which induces an isomorphism $\mathcal{E}_\tau[\pi^{e-1}] \simeq \mathcal{E}_\tau/\mathcal{E}_\tau[\pi]$. One can then define a map $\zeta_\tau^d : \bigwedge^d \mathcal{E}_{\sigma\tau}/\pi \twoheadrightarrow \bigwedge^d \mathcal{E}_\tau^{(p)}/\pi$ for any integer d, which is analogous to the one previously defined (i.e. corresponds to the Verschiebung divided by a power of π). For a

$\mathcal{O}_S \otimes_{\mathbb{Z}_p} O_L$-module $\mathcal{M}$, we write $\mathcal{M}/\pi := \mathcal{M} \otimes_{O_L} (O_L/\pi O_L)$. The section $Ha_\tau^{[j]}$ is thus defined using the composition

$$\begin{array}{ccc} \bigwedge^{d_{\tau,i}} \mathcal{E}_\tau/\pi & \xrightarrow{(\bigwedge^{d_{\tau,i}} V_{\sigma\tau})^{(pf-1)} \circ (\zeta_{\sigma\tau}^{d_{\tau,i}})^{(pf-2)} \circ \dots \circ \zeta_{\sigma^{-1}\tau}^{d_{\tau,i}}} & \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[e]} \mathcal{E}_\tau^{(p^f)}/\pi \\ \bigwedge^{d_{\tau,i}} \frac{1}{\pi^{e-i}} \Big\uparrow & & \Big\downarrow (\bigwedge^{d_{\tau,i+1}} \pi \circ \dots \circ \bigwedge^{d_{\tau,e}} \pi)^{(p^f)} \\ \bigwedge^{d_{\tau,i}} (\pi^{e-i}\mathcal{E}_\tau)/\pi & & \bigwedge^{d_{\tau,i}} \mathrm{Fil}^{[i]} \mathcal{E}_\tau^{(p^f)}/\pi \end{array}$$

Here the map $\bigwedge^{d_{\tau,j}} \pi$ is a map which should correspond to the multiplication by $\pi^{d_{\tau,j}}$.

The main property is that the section ${}^\mu\mathrm{Ha}$, the product of the sections $Ha_\tau^{[i]}$, determines the μ-ordinary locus.

Theorem 2.2.2 ([BH1] Cor. 3.2.3)**.** *Let x be a closed point of S. Then the p-divisible group G_x is μ-ordinary if and only if ${}^\mu\mathrm{Ha}(x) \neq 0$.*

3 Canonical subgroup and canonical filtration

In this section, K denotes a finite extension of $\mathbb{Q}_p$, and G a p-divisible group over its ring of integers O_K oh height h_0 and dimension d_0.
If $w \geq 0$ is a real, we denote $p^w O_K := \{x \in O_K, v(x) \geq w\}$ and $O_{K,\{w\}} := O_K/p^w O_K$.

3.1 The canonical subgroup and its partial degrees

Considering the reduction of G modulo p, on can consider the (classical) Hasse invariant $ha(G)$, and get an element in O_K/p. Taking its valuation, one gets an element $v(ha(G)) \in [0,1]$.
If $C \subseteq G[p]$ is a finite flat subgroup, one defines its degree $\deg C$ as the valuation of the determinant of the map $\omega_{G/C} \to \omega_G$ (see [Fa1] for the definition and properties of the degree).

We recall the result from Fargues on the canonical subgroup.

Theorem 3.1.1 ([Fa2])**.** *Suppose $p \neq 2$, and let G be a p-divisible group of height h_0 and dimension d_0 over O_K. We suppose that $v(ha(G_0)) < 1/2$, and that $v(ha(G_0)) < 1/3$ if $p = 3$. Then there exists a canonical subgroup $C_0 \subseteq G[p]$, such that :*

- *C_0 has height d_0.*
- *$\deg C_0 = d_0 - v(ha(G_0))$.*
- *C_0 is the kernel of the Frobenius in $G \times_{O_K} O_{K,\{1-ha(G_0)\}}$.*
- *if $v(ha(G)) < 1/(p+1)$ then we have $v(ha(G/C_0)) = p \cdot ha(G_0)$.*

One issue is that the previous result is not valid for $p = 2$. Another one is that it constructs a subgroup called the canonical subgroup, but does not give a definition for it. An alternative is to introduce the following notion.

Definition 3.1.2. *Let G be a p-divisible group of height h_0 and dimension d_0 over O_K. Let C_0 be a finite flat subgroup of $G[p]$. We say that C_0 is the canonical subgroup of G if the height of C_0 is d_0 and if $\deg C_0 > d_0 - 1/2$.*

If a canonical subgroup exists, it is necessarily unique, hence the definition. Indeed, one has the following proposition.

Proposition 3.1.3 ([Bi4] Theorem 3.2)**.** *Let G be a p-divisible group of height h_0 and dimension d_0 over O_K. Suppose that there exists a canonical subgroup C_0. Then C_0 is unique. Moreover, we have the relation $\deg C_0 = d_0 - v(ha(G))$, and C_0 is the kernel of the Frobenius in $G \times_{O_K} O_{K,\{1-v(ha(G))\}}$.*

When one considers a p-divisible group G with an action of O_L, one can say more. Indeed, one can define partial degrees for finite flat subgroups of $G[p]$.
Assume that G has an action of O_L. Then one has a decomposition $\omega_G = \oplus_{\tau\in\mathcal{T}}\omega_{G,\tau}$, with $O_{L^{ur}}$ acting by τ on $\omega_{G,\tau}$. Let us fix an ordering on $\Sigma_\tau = \{\sigma_1, \dots, \sigma_e\}$. This gives a filtration

$$0 \subseteq \omega_{G,\tau}^{[1]} \subseteq \dots \subseteq \omega_{G,\tau}^{[e]} = \omega_{G,\tau}$$

Indeed, one has a direct sum in generic fiber $\omega_{G,\tau} \otimes_{O_K} K = \oplus_{i=1}^{e} \omega_{G,\tau,i,K}$, with O_L acting by σ_i on $\omega_{G,\tau,i,K}$. Then one defines for $1 \leq i \leq e$

$$\omega_{G,\tau}^{[i]} := (\bigoplus_{j=1}^{i} \omega_{G,\tau,j,K}) \cap \omega_{G,\tau}$$

The O_K-modules $\omega_{G,\tau}^{[i]}$ are free. Assume that each graded part $\omega_{G,\tau}^{[j]}/\omega_{G,\tau}^{[j-1]}$ has rank d. This implies that there is no obstruction for (the reduction modulo p) of G to be ordinary.

Definition 3.1.4. *Let $C \subseteq G[p]$ be a finite flat subgroup stable by O_L. We define the partial degree $\deg_\tau^{[j]} C$ as the valuation of the map*

$$\omega_{G/C,\tau}^{[j]}/\omega_{G/C,\tau}^{[j-1]} \to \omega_{G,\tau}^{[j]}/\omega_{G,\tau}^{[j-1]}$$

These partial degrees have first been introduced in [Sa2] fot the Hilbert modular variety.
Taking the reduction modulo π of G, one sees that the above filtration gives a PR datum. In particular, one can define the primitive Hasse invariants $hasse_\tau$ and $m_\tau^{[j]}$. These are elements in O_K/π. One can then relate the partial degrees of the canonical subgroup to the primitive Hasse invariants.

Theorem 3.1.5 ([Bi4] Theorem 3.4)**.** *Assume that there exist a canonical subgroup C, and that $v(ha(G)) \leq \min(\frac{1}{2}, \frac{1}{e})$. Then*

$$\deg_\tau^{[1]} C[\pi] = d - v(hasse_\tau) \qquad \deg_\tau^{[j]} C[\pi] = d - v(m_\tau^{[j]})$$

for every $\tau \in \mathcal{T}$ and $2 \leq j \leq e$.

Actually, one can say much more, and compute the partial degrees of $C[\pi^k]$. One can also compute the primitive Hasse invariants for $G/C[\pi]$. We refer to [Bi4] section 3 for more details.

3.2 Canonical filtration in the unramified case

The previous section dealt with the ordinary case, i.e. the classical Hasse invariant and the canonical subgroup. This implied that the ranks of the graded parts $\omega_{G,\tau}^{[j]}/\omega_{G,\tau}^{[j-1]}$ must be constant.

One can of course ask what happens if this hypothesis is removed. In this section, we will deal with the unramified case, i.e. we will only consider the action of $O_{L^{ur}}$. One has the decomposition

$$\omega_G = \oplus_{\tau \in \mathcal{T}} \omega_{G,\tau}$$

where $\omega_{G,\tau}$ is a free O_K-module. Let d_τ be its rank, and let us order these elements $d_1 \leq \dots \leq d_f$.

Definition 3.2.1. *A canonical filtration is a filtration by finite group schemes $0 \subseteq C_1 \subseteq \dots \subseteq C_f \subseteq G[p]$ such that C_i has height fd_i and*

$$\deg C_i \geq \sum_{j=1}^{f} \min(d_j, d_i) - \frac{1}{p+1}$$

This notion generalizes the canonical subgroup in the ordinary case. One has similar properties. Recall that one has construct Hasse invariants Ha_τ in section 2.2, which are elements in O_K/p.

Proposition 3.2.2 ([Bi6] Theorem 4.3.3)**.** *Assume that a canonical filtration exists. Then it is unique. Moreover, let $\tau \in \mathcal{T}$, and let C be the subgroup in the canonical filtration of height fd_τ. Then*

$$\sum_{k=0}^{f-1} p^k \deg_{\sigma^{-k}\tau} C = \sum_{k=0}^{f-1} p^k \min(d_{\sigma^{-k}\tau}, d_\tau) - v(Ha_\tau)$$

One can actually more, and compute the partial degrees $\deg_{\tau'} C$. For this, one needs to introduce the refined partial Hasse invariants ; these elements allow us to factor the section Ha_τ. They do not exist on an arbitrary base, but one needs the existence of an adequate filtration on $\omega_{G,\tau'}$ for each $\tau' \in \mathcal{T}$. We refer to [Bi6] section 1 for the precise definitions of adequate filtrations and refined partial Hasse invariants, and to [Bi6] Theorem 4.3.3 for the following result.

Proposition 3.2.3 ([Bi6] Theorem 4.3.3)**.** *Assume that a canonical filtration exists. Then one can define adequate filtrations on $\omega_{G,\tau}$ and $\omega_{G^D,\tau}$ for every $\tau \in \mathcal{T}$. Thanks to these filtrations, the section Ha_τ can be factored as*

$$Ha_\tau = \prod_{k=0}^{f-1} (ha_\tau^{[k]})^{p^k}$$

If C is the subgroup in the canonical filtration of height fd_τ, then

$$\deg_{\sigma^{-k}\tau} C = \min(d_{\sigma^{-k}\tau}, d_\tau) - v(ha_\tau^{[k]})$$

3.3 Towards a definition in the general case

One does not have a good theory of the canonical filtration in the ramified setting. We suggest here a possible definition, and a conjecture for the link with the μ-ordinary Hasse invariant.
In this section only, we assume that L is totally ramified of degree e. One has a filtration

$$0 \subseteq \omega_G^{[1]} \subseteq \dots \subseteq \omega_G^{[e]} = \omega_G$$

and let d_i be the rank of $\omega_G^{[i]}/\omega_G^{[i-1]}$. Assume that we have chosen the ordering such that $d_1 \geq \dots \geq d_e$. In the previous section, one has constructed the Hasse invariants $Ha^{[1]}, \dots, Ha^{[e]}$.

Definition 3.3.1. *A canonical filtration for G is a filtration $0 \subseteq C_1 \subseteq \cdots \subseteq C_e \subseteq G[p]$ by finite flat group schemes such that*

— *C_i/C_{i-1} has height d_i.*

— *$\pi \cdot C_i \subseteq C_{i-1}$.*

— *$\deg(C_i/C_{i-1}) \geq \sum_{j=1}^{e} \frac{\min(d_i,d_j)}{e} - \frac{1}{2}$.*

One can formulate the following conjecture, which relates the degrees of the subgroups to the μ-ordinary Hasse invariants constructed in section 2.2.

Conjecture 3.3.2. *Assume that a canonical filtration exists. Then each step of the filtration is unique. Moreover, one has the relations*

$$\deg(C_i/C_{i-1}) = \sum_{j=1}^{e} \frac{\min(d_i,d_j)}{e} - v(Ha^{[i]})$$

for every $1 \leq i \leq e$.

4 Geometry of Shimura varieties

In the first section, one studied p-divisible groups over a field, giving thus a local approach. One can have a glocal point of view by studying Shimura varieties, and their geometry.

4.1 Definition of the varieties

4.1.1 Unitary Case

Let F_0 be a totally real field, and F/F_0 be a CM extension. Let O_F be the ring of integers of F, Σ be the set of embeddings of F into $\overline{\mathbb{Q}_p}$; let $(\alpha_\sigma)_{\sigma\in\Sigma}$ be a collection of integers such that the quantity $h := \alpha_\sigma + \alpha_{\overline{\sigma}}$ does not depend on σ, where $\overline{\sigma}$ is the composition of σ with the complex conjugation. Let K_0 be a finite extension of $\mathbb{Q}_p$ containing all the embeddings of F into $\overline{\mathbb{Q}_p}$.

Definition 4.1.1. *Let Y_{naive} be the Shimura variety over O_{K_0} associated to F_0/F and the signature (α_σ). It is a moduli space which S-points are the isomorphism classes of $(A, \lambda, \iota, \eta)$ where*

— *$A \to S$ is an abelian scheme*

— *$\lambda : A \to A^t$ is a prime to p polarization.*

— *$\iota : O_F \to End\ A$ is compatible with complex conjugation and the Rosati involution, and satisfies the Kottiwtz condition.*

— *η is a level structure.*

We refer to [Bi3] section 1.1 or [BH2] section 2.4 for more details about this definition. The Kottwitz condition prescribes the characteristic polynomial of any element $x \in O_F$ acting on the sheaf ω_A, the dual of the Lie algebra of A. More precisely, one requires that the characteristic polynomial of x acting on ω_A should be

$$\prod_{\sigma\in\Sigma} (X - \sigma(x))^{\alpha_\sigma}$$

Let us now consider the sheaf ω_A. One has a decomposition

$$\omega_A = \bigoplus_\pi \omega_{A,\pi}$$

where π runs through the places of F above p. Let us fix such a place π. Let L be the completion of F at π; the sheaf $\omega_{A,\pi}$ has thus an action of O_L. Let L^{ur} be the maximal unramified extension contained in L, and $\mathcal{T}$ the set of embeddings of L^{ur} into $\overline{\mathbb{Q}_p}$. One has a decomposition

$$\omega_{A,\pi} = \bigoplus_{\tau \in \mathcal{T}} \omega_{A,\pi,\tau}$$

One can thus define the PR Shimura variety. Let us fix an ordering $\sigma_1, \dots, \sigma_e$ on the embeddings of L above τ.

Definition 4.1.2. *Let $A \to S$ be an abelian scheme as before. A PR datum for $\omega_{A,\pi,\tau}$ is a filtration*

$$0 \subseteq \omega^{[1]}_{A,\pi,\tau} \subseteq \dots \subseteq \omega^{[e]}_{A,\pi,\tau} = \omega_{A,\pi,\tau}$$

such that
- *$\omega^{[i]}_{A,\pi,\tau}$ is locally a direct factor, and is stable by O_L.*
- *O_L acts by σ_i on $\omega^{[i]}_{A,\pi,\tau}/\omega^{[i-1]}_{A,\pi,\tau}$.*
- *$\omega^{[i]}_{A,\pi,\tau}/\omega^{[i-1]}_{A,\pi,\tau}$ is locally free over $\mathcal{O}_S$ of rank α_{σ_i}.*

Definition 4.1.3. *Let us define Y_{PR} as the variety over O_{K_0} classifying $(A, \lambda, \iota, \eta)$ in Y_{naive} with a PR datum on $\omega_{A,\pi,\tau}$ for all π and τ. One also requires that the PR datum is compatible with the polarization.*

We refer to [BH2] Def. 2.21 for the precise definition. Let us be more precise about the compatibility with the polarization. First of all, let us remark that a PR datum for $\omega_{A,\pi,\tau}$ naturally induces one for $\omega_{A^t,\pi,\tau}$. Indeed, if $\mathcal{E} = H^1_{dR}(A)$, one has a decomposition $\mathcal{E} = \oplus_\pi \mathcal{E}_\pi$, and $\mathcal{E}_\pi = \oplus_{\tau \in \mathcal{T}} \mathcal{E}_{\pi,\tau}$. The Hodge filtration allows us to see $\omega_{A,\pi,\tau}$ as a subsheaf of $\mathcal{E}_{\pi,\tau}$. Then can then complete the PR datum

$$0 \subseteq \omega^{[1]}_{A,\pi,\tau} \subseteq \dots \subseteq \omega^{[e]}_{A,\pi,\tau} = \omega_{A,\pi,\tau} \subseteq \omega^{[e+1]}_{A,\pi,\tau} \subseteq \dots \subseteq \omega^{[2e]}_{A,\pi,\tau} = \mathcal{E}_{\pi,\tau}$$

To see this, let us consider an uniformizer π in L, and let $\pi_1, \dots, \pi_e$ be the conjugates of π in L, and let $E(X) = \prod_{i=1}^e (X - \pi_i)$ be the minimal polynomial of π. Then $\mathcal{E}_{\pi,\tau}$ is locally free over $\mathcal{O}_S[X]/E(X)$, with X acting by π. One then defines

$$\omega^{[e+l]}_{A,\pi,\tau} := Q_l(\pi)^{-1} \omega^{[e-l]}_{A,\pi,\tau}$$

with $Q_l(X) = \prod_{i=e-l+1}^e (X - \pi_i)$. Note that the multiplication by $Q_l(\pi)$ induces an isomorphism between $\mathcal{E}_{\pi,\tau}/\mathcal{E}_{\pi,\tau}[Q_l(\pi)]$ and $\mathcal{E}_{\pi,\tau}[Q^l(\pi)]$, with $Q^l(X) = \prod_{i=1}^{e-l} (X - \pi_i)$. Since $\omega^{[e-l]}_{A,\pi,\tau}$ is a locally free sheaf inside $\mathcal{E}_{\pi,\tau}[Q^l(\pi)]$, this justifies the definition of $\omega^{[e+l]}_{A,\pi,\tau}$.
Since $(\mathcal{E}_{\pi,\tau}/\omega_{A,\pi,\tau})^\vee$ is isomorphic to $\omega_{A^t,\pi,\tau}$, one gets the desired filtration on this sheaf.

Now let us distinguish several cases. Let π_0 be the prime of F_0 below π.

- Assume that π_0 splits in F as $\pi_0 = \pi \cdot \pi'$. Then the polarization induces an isomorphism between $\mathcal{E}_\pi$ and $\mathcal{E}_{\pi'}$. We then ask that the complete filtrations on $\mathcal{E}_{\pi,\tau}$ and $\mathcal{E}_{\pi',\tau}$ are compatible with the polarization. We refer to this case as the (AL) case.
- Assume that π_0 is inert in F. Then the complex conjugation acts on $\mathcal{T}$, and the polarization induces an isomorphism between $\mathcal{E}_{\pi,\tau}$ and $\mathcal{E}_{\pi,\overline{\tau}}$. We then require that the complete filtrations on these sheaves are compatible with the polarization. We refer to this case as the (AU) case.
- The case where π_0 ramifies in F is more involved. We refer to this as the (AR) case, and will be dealt with in a future section.

4.1.2 Hilbert-Siegel varieties

Recall that F_0 denotes a totally real field, and let $g \geq 1$ be an integer.

Definition 4.1.4. *Let $\mathcal{A}_g$ be the Hilbert-Siegel variety over O_{K_0} associated to F_0. It is a moduli space which S-points are the isomorphism classes of $(A, \lambda, \iota, \eta)$ where*
- *$A \to S$ is an abelian scheme*
- *$\lambda : A \to A^t$ is a prime to p polarization.*
- *$\iota : O_{F_0} \to End\ A$ satisfies the Kottiwtz condition.*
- *η is a level structure.*

The Kottwitz condition prescribes the characteristic polynomial of any element $x \in O_{F_0}$ acting on the sheaf ω_A, the dual of the Lie algebra of A. More precisely, one requires that the characteristic polynomial of x acting on ω_A should be

$$\prod_{\sigma \in \Sigma} (X - \sigma(x))^g$$

This is the naive integral model, and one can similarly define an integral model thanks to PR data. The process is similar as the previous case.
The sheaf ω_A decomposes as $\oplus_{\pi_0} \omega_{A,\pi_0}$, where π_0 runs through the places of F_0 above p. Let us fix such a place π_0. Let L be the completion of F_0 at π_0 ; the sheaf ω_{A,π_0} has thus an action of O_L. Let L^{ur} be the maximal unramified extension contained in L, and $\mathcal{T}$ the set of embeddings of L^{ur} into $\overline{\mathbb{Q}_p}$. One has a decomposition

$$\omega_{A,\pi_0} = \bigoplus_{\tau \in \mathcal{T}} \omega_{A,\pi_0,\tau}$$

One can thus define the PR Shimura variety. Let us fix an ordering $\sigma_1, \dots, \sigma_e$ on the embeddings of L above τ.

Definition 4.1.5. *Let $A \to S$ be an abelian scheme as before. A PR datum for $\omega_{A,\pi_0,\tau}$ is a filtration*

$$0 \subseteq \omega^{[1]}_{A,\pi_0,\tau} \subseteq \dots \subseteq \omega^{[e]}_{A,\pi_0,\tau} = \omega_{A,\pi_0,\tau}$$

such that
- *$\omega^{[i]}_{A,\pi_0,\tau}$ is locally a direct factor, and is stable by O_L.*
- *O_L acts by σ_i on $\omega^{[i]}_{A,\pi_0,\tau}/\omega^{[i-1]}_{A,\pi_0,\tau}$.*
- *$\omega^{[i]}_{A,\pi_0,\tau}/\omega^{[i-1]}_{A,\pi_0,\tau}$ is locally free over $\mathcal{O}_S$ of rank g.*

Definition 4.1.6. *Let us define $\mathcal{A}_{g,PR}$ as the variety over O_{K_0} classifying $(A, \lambda, \iota, \eta)$ in $\mathcal{A}_g$ with a PR datum on $\omega_{A,\pi_0,\tau}$ for all π_0 and τ. One also requires that the PR datum is compatible with the polarization.*

We refer to [BH2] Def. 2.21 for the precise definition. The condition about the polarization is simpler in this situation. Let $\mathcal{E} = H^1_{dR}(A)$; one has a decomposition $\mathcal{E} = \oplus_{\pi_0}\mathcal{E}_{\pi_0}$, and $\mathcal{E}_{\pi_0} = \oplus_{\tau\in\mathcal{T}}\mathcal{E}_{\pi_0,\tau}$. The Hodge filtration implies that $\omega_{A,\pi_0,\tau}$ is locally a direct factor of $\mathcal{E}_{\pi_0,\tau}$. One can then complete the PR datum into a complete filtration as in the previous section

$$0 \subseteq \omega^{[1]}_{A,\pi_0,\tau} \subseteq \cdots \subseteq \omega^{[e]}_{A,\pi_0,\tau} = \omega_{A,\pi_0,\tau} \subseteq \omega^{[e+1]}_{A,\pi_0,\tau} \subseteq \cdots \subseteq \omega^{[2e]}_{A,\pi_0,\tau} = \mathcal{E}_{\pi_0,\tau}$$

The compatibility with the polarization requires that this complete filtration is equal to its orthogonal, i.e. $\omega^{[e+i]}_{A,\pi_0,\tau} = {\omega^{[e-i]}_{A,\pi_0,\tau}}^{\perp}$.

Remark 4.1.7. *One has a non degenerate alternating pairing $<,>$ on $\mathcal{E}_{\pi_0,\tau}$, and $\omega_{A,\pi_0,\tau}$ is totally isotropic for that pairing. One has the inclusion $\omega^{[1]}_{A,\pi_0,\tau} \subseteq \mathcal{E}_{\pi_0,\tau}[\pi-\pi_1]$, and one has a modified pairing on this space. Recall that $\mathcal{E}_{\pi_0,\tau}[\pi-\pi_1] = Q_1(\pi)\mathcal{E}_{\pi_0,\tau}$; the modified pairing is given locally by*

$$\{Q_1(\pi)x, Q_1(\pi)y\} := < x, Q_1(\pi)y >$$

The above condition implies that $\omega^{[1]}_{A,\pi_0,\tau}$ is totally isotropic for this alternating pairing.
More generally, it implies that $\omega^{[i]}_{A,\pi_0,\tau}/\omega^{[i-1]}_{A,\pi_0,\tau}$ is totally isotropic for the modified pairing on the sheaf

$$(\mathcal{E}_{\pi_0,\tau}/\omega^{[i-1]}_{A,\pi_0,\tau})[\pi-\pi_i]$$

4.2 Density results

The generic fibers of the spaces $A_{g,PR}$ and Y_{PR} are the same as the ones for the naive models. One can then be interested in the geometry of the special fiber. Let $A_{g,PR,s}$ and $Y_{PR,s}$ be the special fibers of these spaces respectively. These are k-scheme, where k is the residue field of K_0.

Theorem 4.2.1 ([BH2] Th. 2.30)**.** *Assume that no prime above p falls in the (AR) case. Then the schemes $A_{g,PR,s}$ and $Y_{PR,s}$ are smooth.*

Let us fix a prime π_0 of F_0, not in the (AR) case. One thus has integers $(d_{\tau,i})_{\tau\in\mathcal{T},1\leq i\leq e}$, and a p-divisible group G with a PR datum of type $(d_\bullet)$.
Let x be a closed point of one of these spaces. Then one can define the polygons $\mathrm{Newt}_{O_L}(x)$ and $\mathrm{Hdg}_{O_L}(x)$, following 1.1. The generealized Rapoport locus (resp. μ-ordinary locus) is the locus where the polygon $\mathrm{Hdg}_{O_L}(x)$ (resp. $\mathrm{Newt}_{O_L}(x)$) is equal to the polygon $PR(d_\bullet)$.

Theorem 4.2.2 ([BH2] Th. 3.3 and 4.1)**.** *The generalized Rapoport locus, and the μ-ordinary locus are dense.*

For the space $A_{g,PR,s}$, the μ-ordinary locus coincides with the usual ordinary locus. For the space $Y_{PR,s}$, one has the following criterion.

Proposition 4.2.3 ([BH2] Prop. 4.26)**.** *The ordinary locus (at π) of $Y_{PR,s}$ is non empty if and only if and only if there exists an integer a such that*

$$\alpha_\sigma = a$$

for all $\sigma \in \Sigma$ above π.

The last condition is also equivalent to the fact that the local reflex field is equal to $\mathbb{Q}_p$.

4.3 The Hodge stratification for $e \leq 3$

One would hope to say more about the geometry of the spaces $A_{g,PR,s}$ and $Y_{PR,s}$; let us denote by Y one of these spaces. More precisely, one could try to look at the variation of the polygons $Newt_{O_L}$ and Hdg_{O_L}, and hope that these would define a nice stratification on the variety.
Looking at the Hodge polygon might seem easier, but even then the situation is not as good as one would expect. A naive approach consist in looking at the Hodge polygon, which gives a description

$$Y = \coprod_P Y_P$$

where P runs through the set of all possible polygons, and Y_P consists in all the points x whose Hodge polygon is P. One can see that this defines a weak stratification, in the sense that the closure of a strata is included in a union of strata (since the Hodge polygon goes up by specialization). However, if $e \geq 3$, this is not a strong stratification : this closure is in general not equal to a union of strata.
One can then try to consider the isomorphism class of the PR datum. However, for large ramification index e, this does not give a finite number of strata.
One can be more precise when $e \leq 3$. When $e = 1$, there is nothing to say.
When $e = 2$, the Hodge polygon is determined by one integer, the dimension of the π torsion of the sheaf. This integer determines a strong stratification.
Assume now that $e = 3$. Let τ be an element of $\mathcal{T}$, and let us consider the sheaf $\omega_{A,\pi_0,\tau}$ that we write ω_τ for simplicity. One has a filtration

$$0 \subseteq \omega_\tau^{[1]} \subseteq \omega_\tau^{[2]} \subseteq \omega_\tau$$

and let us write $d_1, d_1 + d_2, d_1 + d_2 + d_3$ the rank of these sheaves. The idea is to consider not only the polygon $\mathrm{Hdg}(\omega_\tau)$, but the polygons $\mathrm{Hdg}(\omega_\tau/\omega_\tau^{[1]})$ and $\mathrm{Hdg}(\omega_\tau^{[2]})$ as well. These polygons are defined at each point of the variety as follows : if M is a k vector space, which is a $K[X]/X^j$-module for some integer j, the polygon $\mathrm{Hdg}(M)$ classifies the structure of M as a $k[X]/X^j$-module. For example, if $j = 1$, this polygon is determined by the dimension of M.
Let k be a field, and let C be the set of isomorphism classes of filtrations $0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}$ where $\mathcal{F}$ is a $k[X]/X^3$-module, which has dimension $d_1 + d_2 + d_3$ over k, and $\mathcal{F}_1, \mathcal{F}_2$ is a PR datum for the integers (d_1, d_2, d_3).

Proposition 4.3.1 ([Bi7] Th. 3.5). *The set C is finite, independent of the field k, and an element of C is determined by the three polygons* $\mathrm{Hdg}(\mathcal{F})$, $\mathrm{Hdg}(\mathcal{F}/\mathcal{F}_1)$ *and* $\mathrm{Hdg}(\mathcal{F}_2)$.

One can also describe the exact conditions that are imposed on these three polygons. Moreover, one has a natural order on the set of polygons, hence on C.
For $c \in C$, we denote by Y_c the set of points of Y such that the induced filtration is of class c.

Proposition 4.3.2 ([Bi7] Th. 4.5 and Th. 5.3). *One has a strong stratification*

$$Y = \coprod_{c \in C} Y_c$$

By a strong stratification, we mean that the closure of Y_c is equal to

$$\coprod_{c' \geq c} Y_{c'}$$

for all $c \in C$.

4.4 The (AR) case

In the previous sections, we excluded the case where a prime in F_0 ramifies in F. We study this situation into more details here.
For simplicity, assume in this section only that $F_0 = \mathbb{Q}$. We also assume that $p \neq 2$. Thus F is a quadratic imaginary extension, and assume that p ramifies in F. Let F_p be the completion of F at p, and π a uniformizer of F_p. We write σ_1, σ_2 the embeddings of F_p into $\overline{\mathbb{Q}_p}$, and let us define $\pi_i := \sigma_i(\pi)$. Let a, b be integers with $a \leq b$, and define $h = a + b$.

Definition 4.4.1. *Let Y_0 be the moduli space over O_F whose R-points are couples $(A, \lambda, \iota, \eta, \omega_1)$, where*

- *— A is an abelian scheme over R of dimension h*
- *— λ is a polarization*
- *— $\iota : O_F \to End(A)$, making the Rosati involution and the complex conjugation compatible*
- *— η is a level structure*
- *— $\omega^{[1]} \subseteq \omega_A$ is a locally direct factor of rank a, stable by O_F*
- *— O_F acts by σ_1 on $\omega^{[1]}$, and by σ_2 on $\omega_A/\omega^{[1]}$.*

Let $\mathcal{E} = H^1_{dR}(A)$; it is a locally free sheaf on Y_0 of rank $2h$. If has an action of O_F, and is locally free of rank h over $\mathcal{O}_{Y_0} \otimes_{\mathbb{Z}} O_F$. The Hodge filtration is $\omega_A \subseteq \mathcal{E}$. The sheaf $\mathcal{E}$ has an action of O_{F_p}, and let $[u]$ be the action of u on $\mathcal{E}$ for every $u \in O_{F_p}$. The last condition implies that $([\pi] - \pi_1) \cdot \omega_1 = 0$ and $([\pi] - \pi_2) \cdot \omega \subseteq \omega_1$.

Thanks to the polarization, one has a perfect alternating pairing on $<,>$ on $\mathcal{E}$. Let us define $\omega^{[2]} \subseteq \mathcal{E}$ by the formula

$$\omega^{[2]} = (([\pi] - \pi_2)^{-1}\omega^{[1]})^{\perp}$$

The sheaf $\omega^{[2]}$ is locally free of rank b, and one has $\omega^{[2]} \subseteq \omega$. Moreover, one has

$$([\pi] - \pi_2) \cdot \omega^{[2]} = 0 \qquad ([\pi] - \pi_1) \cdot \omega \subseteq \omega^{[2]}$$

Let Y_s be the special fiber of Y_0. We are interested in the geometry of Y_s. First let us introduce some functions on Y_s.

Definition 4.4.2. *Let k be a field of characteristic p, and let $x \in Y_s(k)$. Let us define the integers $(h(x), l(x))$ as the dimension of $\pi \cdot \omega$, and $\omega^{[1]} \cap \omega^{[2]}$ respectively.*

One can prove that one has the inequalities

$$0 \leq h(x) \leq l(x) \leq a$$

One then has a stratification on Y_s

$$Y_s = \coprod_{0 \leq h \leq l \leq a} Y_{s,h,l}$$

where $Y_{s,h,l}$ consists of the points x with $(h(x), l(x)) = (h, l)$. One has the following result regarding the geometry of Y_s.

Theorem 4.4.3 ([BH3] Prop. 1.10 and 1.12)**.** *Let (h, l) be integers with $0 \leq h \leq l \leq a$, and let $\overline{Y_{s,h,l}}$ be the closure of $Y_{s,h,l}$. Then*

$$\overline{Y_{s,h,l}} = \coprod_{0 \leq h' \leq h \leq l \leq l' \leq a} Y_{s,h',l'}$$

Moreover, the stratum $Y_{s,h,l}$ is nonempty, and is equidimensional of dimension $ab - \frac{(l-h)(l-h+1)}{2}$. The smooth locus of Y_s is the union of the strata $Y_{s,h,h}$ for $0 \leq h \leq a$.

Remark 4.4.4. *In special fiber, the pairing $<,>$ induces a modified pairing on $\mathcal{E}[\pi]$ by the formula*

$$\{\pi x, \pi y\} := < \pi x, y >$$

The main difference in this case is that this pairing is symmetric, contrary to the case of the Hilbert-Siegel variety.
The sheaf $\omega^{[2]}$ is the orthogonal of $\omega^{[1]}$ for this pairing.

In particular, the stratum $X_{0,a}$ is closed and the strata $X_{h,h}$ are open, for every $0 \leq h \leq a$.

5 Applications to overconvergent modular forms

The theory of overconvergent modular forms has been developed by Katz, Coleman and many others. The idea is to study section of modular sheaves on a special locus, and not the whole variety. First, one must consider the rigid analytic variety associated to the modular curve. One can then consider the ordinary locus. By definition, an overconvergent modular form is a section on a strict neighborhood of the ordinary locus.
If one considers a classical weight, the modular sheaves is already defined on the variety. The important part of the theory is to be able to consider p-adic weights ; they can be defined on the ordinary locus using the Igusa tower. To define them on a strict neighborhood is a more involved question.
If one considers Shimura vareties with non-empty ordinary locus, one can try to generalize this definition. An important result about overconvergent modular form is a classicality result : an overconvergent modular form, which is an eigenvector for some Hecke operators is classical, provided that the valuation of the eigenvalue is small enough.

As was seen in the previous sections, some Shimura varieties have an empty oridnary locus, and one might think that the theory of overconvergent modular forms should fail in this situation. However, one can consider instead the μ-ordinary locus, which is dense in the special fiber.

Let us consider an unitary Shimura variety.

5.1 The unramified case

In this section, assume that the CM field F is unramified at p. We can then consider the scheme Y_{naive} defined in the previous section. Let $\mathcal{Y}$ be the rigid analytic space associated to it ; one has

an admissible open subset $\mathcal{Y}^{\mu-ord}$ defined by the μ-ordinary locus. To define this locus, one can use a lift of the μ-ordinary Hasse invariant.
Alternatively, one can work at the Iwahori level. Recall that π is a place of F above p; one can then consider the p-divisible group $A[\pi^\infty]$ on the variety. For each embedding τ above π, let d_τ be the dimension of ω_τ, the τ-part of the sheaf of differentials of the above p-divisible group. We write $d_1 < \cdots < d_r$ these integers.

Definition 5.1.1. *Let $Y_{naive,Iw}$ be the scheme over Y_{naive} parametrizing subgroups $H_1 \subseteq \cdots \subset H_r \subseteq A[\pi]$, where H_i is a finite flat subgroup of height d_i, stable by O_F.*

In case (AU) one requires moreover that the chain of subgroups is stable under duality.
Let $\mathcal{Y}_{Iw}$ be the associated rigid space. We define $\mathcal{Y}_{Iw}^{mult}$ as the locus where the subgroups H_i have maximal degree, i.e. $\deg H_i = \sum_{\tau \in \mathcal{T}} \min(d_i, d_\tau)$.

Proposition 5.1.2 ([Bi3] Prop. 1.34)**.** *The natural projection $\mathcal{Y}_{Iw} \to \mathcal{Y}$ induces a bijection between $\mathcal{Y}_{Iw}^{mult}$ and $\mathcal{Y}^{\mu-ord}$.*

This observation leads to the following definition of overconvergent modular forms (of classical weight) in this context.

Definition 5.1.3. *Let κ be a classical weight, and let ω^κ be the associated vector bundle. The set of overconvergent modular forms of weight κ is*

$$colim H^0(\mathcal{V}, \omega^\kappa)$$

where $\mathcal{V}$ runs through the strict neighborhoods of $\mathcal{Y}_{Iw}^{mult}$ in $\mathcal{Y}_{Iw}$.

One has the following Hecke operators.

Definition 5.1.4. *Let $U_{\pi,i}$ be the Hecke operator obtained by classifying subgroups $L \subset A[\pi]$ such that $A[\pi] = H_i \oplus L$ in case (AL).*
Let $U_{\pi,i}$ be the Hecke operator obtained by classifying subgroups $L \subset A[\pi^2]$ such that $A[\pi] = H_i \oplus L[\pi] = H_i^\perp \oplus \pi L$ in case (AU).

The Hecke operators are defined on the geometric fibers of $Y_{naive,Iw}$, and preserve the integral structure. They then act on the modular forms on $\mathcal{Y}_{Iw}$, and on the overconvergent modular forms. We refer to [Bi3] section 2.3 for more details.
One has the following classicality result.

Theorem 5.1.5 ([Bi3] Th. 3.17)**.** *Let f be an overconvergent modular form of wight κ, which is an eigenform for the Hecke operators $U_{\pi,i}$ with eigenvalue α_i. If the valuations of the α_i are small enough, then f is a classical modular form.*

5.2 Towards a definition in the general case

Many obstacles must be overcome to develop a theory of overconvergent modular forms in the ramified setting. We present here a possible approach.
Let us now allow ramification in F. For simplicity, assume that $p = \pi_0^e$ is totally ramified in F_0, with ramification index e, and that π_0 splits as $\pi_0 = \pi\pi'$ in F (we are then in the (AL) case). In section 4.1.1 , one has introduced the spaces Y_{PR} and studied their geometry. One also has defined

the notion of a canonical filtration for p-divisible groups in this setting in section 3.3.
As in the previous section, one considers the p-divisible group $A[\pi^\infty]$; it satisfies a PR datum for the integers $(d_\sigma)_{\sigma\in\Sigma}$. Let us order these elements $d_1 \leq \cdots \leq d_e$ these integers.
One can consider the following Iwahori variety.

Definition 5.2.1. *Let $Y_{PR,Iw}$ be the scheme over Y_{PR} parametrizing subgroups $H_1 \subseteq \cdots \subset H_e \subseteq A[\pi^e]$, where H_i/H_{i-1} is a finite flat subgroup of height d_i inside $(A/H_{i-1})[\pi]$.*

Let $\mathcal{Y}_{Iw}$ be the associated rigid space. We define $\mathcal{Y}_{Iw}^{mult}$ where the subgroups H_i have maximal degree, i.e. $\deg(H_i/H_{i-1}) = \frac{1}{e}\sum_{j=1}^{e} \min(d_i, d_j)$ for $1 \leq i \leq e$.

Conjecture 5.2.2. *The natural projection $\mathcal{Y}_{Iw} \to \mathcal{Y}$ induces a bijection between $\mathcal{Y}_{Iw}^{mult}$ and $\mathcal{Y}^{\mu-ord}$, the μ-ordinary locus of the rigid space associated to Y_{PR}.*

This observation leads to the following definition of overconvergent modular forms (of classical weight) in this context.

Definition 5.2.3. *Let κ be a classical weight, and let ω^κ be the associated vector bundle. The set of overconvergent modular forms of weight κ is*

$$colim H^0(\mathcal{V}, \omega^\kappa)$$

where $\mathcal{V}$ runs through the strict neighborhoods of $\mathcal{Y}_{Iw}^{mult}$ in $\mathcal{Y}_{Iw}$.

Regarding the Hecke operators, it might be simpler to consider only one operator, which would correspond to the product of the Hecke operators in the previous section.

Definition 5.2.4. *Let U be the Hecke operator obtained by classifying subgroups $L_1 \subseteq \cdots \subseteq L_e \subset A[\pi^e]$ such that $(A/L_{i-1})[\pi] = (H_i + L_{i-1})/L_{i-1} \oplus L_i/L_{i-1}$.*

The Hecke operators are defined on the geometric fibers of $Y_{PR,Iw}$, and preserve the integral structure. They then act on the modular forms on $\mathcal{Y}_{Iw}$, and on the overconvergent modular forms. One has the following classicality conjecture.

Conjecture 5.2.5. *Let f be an overconvergent modular form of wight κ, which is an eigenform for the Hecke operators $U_{\pi,i}$ with eigenvalue α_i. If the valuations of the α_i are small enough, then f is a classical modular form.*

Presented articles

[Bi3] S. Bijakowski, *Analytic continuation on Shimura varieties with μ-ordinary locus* Algebra Number Theory 10(4) : 843-885 (2016).

[Bi4] S. Bijakowski, *Partial Hasse invariants, partial degrees and the canonical subgroup*, Canadian Journal of Mathematics, 70(4), 742-772.

[Bi5] S. Bijakowski, *Compatibility with duality for partial Hasse invariants* preprint (2016).

[Bi6] S. Bijakowski, *Duality, refined partial Hasse invariants and the canonical filtration*, Math. Res. Let., 25(4), 1109-1142 (2018).

[Bi7] S. Bijakowski, *Hodge stratification in low ramification* preprint (2022).

[BH1] S. Bijakowski and V. Hernandez, *Groupes p-divisibles avec condition de Pappas-Rapoport et invariants de Hasse* Journal de l'École polytechnique-Mathématiques, Tome 4 (2017), pp. 935-972.

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[Bi3] S. Bijakowski, *Analytic continuation on Shimura varieties with μ-ordinary locus* Algebra Number Theory 10(4) : 843-885 (2016).

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