Deformation rings

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Introduction

We want to prove the following theorem :

Theorem 0.1. *Let $p \geq 3$ be a prime, and $\rho : G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Z}_p})$ be a continuous representation, unramified outside a finite set of primes Σ with $p \notin \Sigma$. Suppose that*

- *ρ is odd, of finite image, and of projective image A_5.*
- *$\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}}$ is irreducible.*
- *$\overline{\rho}|_{G_{\mathbb{Q}_v}} = 1$ for $v \in \Sigma_p = \Sigma \cup \{p\}$.*
- *$\overline{\rho}$ is modular.*

Then ρ is modular.

To prove this theorem, one will prove a "$R = T$" theorem, that is to say that R the deformation ring of $\overline{\rho}$ is equal to T, the Hecke algebra. Then ρ, which is a point of R, will correspond to a point of T, and thus comes from a modular form.

In the rest of the paper, we will fix E a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of interger, $\mathbb{F}$ its residual field, π an unifomizer and $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ a representation satisfying the hypothesis of the theorem. We will denote by ψ the determinant of ρ.

1 Local deformation rings at p

Let $\mathcal{A}$ be the category of local artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$, and $\mathbb{D}_p^{\square} : \mathcal{A} \to$ Sets be the functor which assigns to $A \in \mathcal{A}$ the set of framed deformations of $\overline{\rho}|_{G_{\mathbb{Q}_p}} = 1$ with determinant ψ. More precisely, an element of $\mathbb{D}_p^{\square}(A)$ is a representation $\rho_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : G_{\mathbb{Q}_p} \to GL_2(A)$, with $\overline{\rho_0} = 1$ ($\overline{\rho_0}$ is the reduction of ρ_0 modulo the maximal ideal of A), and $\det \rho_0 = \psi|_{G_{\mathbb{Q}_p}}$.
Two such representations $\rho_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\rho_0' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ are equivalent if $\rho_0 = \rho_0'$ (that is why we are talking about framed representations).

Proposition 1.1. *The functor $\mathbb{D}_p^{\square}$ is represented by a ring $R_p^{\square}$.*

Remark 1.2. *Since $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is not absolutely irreducible, we have to take framed representations to ensure representability.*

There is therefore an universal representation $\rho^{univ} : G_{\mathbb{Q}_p} \to GL_2(R_p^{\square})$.

Definition 1.3. *Let $\mathbb{D}_p^{\triangle}$ be the functor from $\mathcal{A}$ to Sets, which assigns to $A \in \mathcal{A}$ the set*

$$\left\{\rho_0 \in \mathbb{D}_p^{\square}(A), \exists \text{ line } \mathcal{L} \text{ stable by } G_{\mathbb{Q}_p} \text{ such that } I_{\mathbb{Q}_p} \text{ acts trivially on } \mathcal{L}\right\}$$

Proposition 1.4. *The functor $\mathbb{D}_p^{\triangle}$ is represented by a ring $R_p^{\triangle}$.*

Again, we have an universal representation $G_{\mathbb{Q}_p} \to GL_2(R_p^{\triangle})$, still denoted by ρ^{univ}, and an universal line $\mathcal{L}^{univ}$ in $R_p^{\triangle}$, stable by $G_{\mathbb{Q}_p}$ with the inertia acting trivially.
An element $\rho_0 \in \mathbb{D}_p^{\triangle}(A)$ (for $A \in \mathcal{A}$) is conjugated to a representation of the form $\begin{pmatrix} \varphi_1 & b \\ 0 & \varphi_2 \end{pmatrix}$ with

- φ_1 an unramified character
- $\varphi_2 = \psi|_{G_{\mathbb{Q}_p}} \varphi_1^{-1}$
- $b \in \varphi_2 \cdot Z^1\left(G_{\mathbb{Q}_p}, \frac{\varphi_1}{\varphi_2}\right)$

Let $s \in G_{\mathbb{Q}_p}$ be an element lifting the Frobenius. We will define a cover of the space $D_p^{\triangle}[1/p] = \text{Spec } R_p^{\triangle}[1/p]$, following Taylor.

Definition 1.5. *Let $R_p^{\triangle, U_p}[1/p]$ be the ring defined by*

$$R_p^{\triangle}[1/p][U_p]/\left(U_p^2 - Tr\rho^{univ}(s)U_p + \psi(s), \rho^{univ}(ts) = \psi(s)U_p^{-1}(\rho^{univ}(t) - 1) + \rho^{univ}(s)\ \forall t \in I_{\mathbb{Q}_p}\right)$$

Note that the last conditions can be rewritten $(\rho^{univ}(t) - 1)(\rho^{univ}(s) - \psi(s)U_p^{-1}) = 0$ for all t in the inertia subgroup.
We will note $D_p^{\triangle, U_p}[1/p] = \text{Spec } R_p^{\triangle, U_p}[1/p]$, and f the map $D_p^{\triangle, U_p}[1/p] \to D_p^{\triangle}[1/p]$.

Proposition 1.6. *The map f is generically an isomorphism.*

Proof. We will compute the fiber of f at a point x of $D_p^{\triangle}[1/p]$. Then ρ_x, the specialization of ρ^{univ} at x, is conjugated to $\begin{pmatrix} \varphi_1 & b \\ 0 & \varphi_2 \end{pmatrix}$ with $\varphi_1\varphi_2 = \psi|_{G_{\mathbb{Q}_p}}$ and φ_1 unramified.
<u>Case 1 :</u> Suppose that $\rho_x|_{I_{\mathbb{Q}_p}} = 1$. Then

$$f^{-1}(x) = \text{ Spec } k(x)[U_p]/(U_p^2 - (\varphi_1(s) + \varphi_2(s))U_p + \psi(s))$$

Over x, f is an etale cover of degree 2 if $\varphi_1(s) \neq \varphi_2(s)$, and a ramified map of degree 2 otherwise.
<u>Case 2 :</u> Suppose that $\rho_x|_{I_{\mathbb{Q}_p}} \neq 1$. Then, up to a change of basis, we can assume that

$$\rho_x|_{I_{\mathbb{Q}_p}} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

with $b \neq 0$. If t is an element of the inertia subgroup with $b(t) \neq 0$, then the kernel of $\rho_x(t) - 1$ is exactly the line generated by e_1, the first vector of the chosen basis. Then the equation $(\rho_x(t) - 1)(\rho_x(s) - \psi(s)U_p^{-1}) = 0$ implies $U_p = \varphi_1(s)$. The map f is thus an isomorphism over x.
To prove the proposition, we will show that if x is a closed point in case 1, there is a generization $\tilde{x}$ of x which is in case 2. If x is a closed point in case 1, we can assume that $\rho_x = \begin{pmatrix} \varphi_1 & b \\ 0 & \varphi_2 \end{pmatrix}$ with

$b = 0$ on the inertia subgroup. The point x correspond to an $\mathcal{O}'$-point of $D_p^{\triangle} = \mathrm{Spec}\ R_p^{\triangle}$. Let b' be a ramified cocycle for φ_1/φ_2 (it exists for dimensionnal reasons), and consider the representation defined over $\mathcal{O}'[[X]]$ by $\rho_{\tilde{x}} = \begin{pmatrix} \varphi_1 & b + X\varphi_2 b' \\ 0 & \varphi_2 \end{pmatrix}$. This gives a point $\tilde{x}$ of $D_p^{\triangle}[1/p]$ which is a generization of x, since the reduction of $\rho_{\tilde{x}}$ modulo X is equal to ρ_x. The fact that $\tilde{x}$ is in case 2 follows from b' being ramified. □

Remark 1.7. *The fact that the map f is of degree 2 in the unramified case corresponds (via an $R = T$ theorem) to the existence of two companion forms.*

2 Local deformation rings at Taylor-Wiles primes

A prime l is said to be a Taylor-Wiles prime if

- $l \notin \Sigma \cup \{p\}$
- $l \equiv 1\ (p)$
- $\overline{\rho}(Frob_l)$ has two distinct eigenvalues $\overline{\alpha_l}$ et $\overline{\beta_l}$.

Let l be a Taylor-Wiles prime, and $\mathbb{D}_l : \mathcal{A} \to \mathrm{Sets}$ the functor of deformations of $\overline{\rho}|_{G_{\mathbb{Q}_l}}$ with determinant ψ. An element of $\mathbb{D}_l(A)$ for $A \in \mathcal{A}$ is then a representation $\rho_0 : G_{\mathbb{Q}_l} \to GL_2(A)$ lifting $\overline{\rho}|_{G_{\mathbb{Q}_l}}$, with $\det \rho_0 = \psi|_{G_{\mathbb{Q}_l}}$, and two such representations ρ_0 and ρ_0' are equivalent if there exists an element $h \in GL_2(A)$, congruent to 1 modulo the maximal ideal of A, with $\rho_0 = h\rho_0' h^{-1}$.

Proposition 2.1. *The functor $\mathbb{D}_l$ is represented by a ring R_l.*

The deformations of $\overline{\rho}|_{G_{\mathbb{Q}_l}}$ are actually very simple.

Proposition 2.2. *Let ρ_0 be an element of $\mathbb{D}_l(A)$, with $A \in \mathcal{A}$. Then ρ_0 is conjugated to a matrix of the form $\begin{pmatrix} \alpha_l & 0 \\ 0 & \beta_l \end{pmatrix}$ with α_l and β_l two tamely ramified characters of $G_{\mathbb{Q}_l}$ lifting respectively $\overline{\alpha_l}$ and $\overline{\beta_l}$.*

Proof. Since $\overline{\rho_0}$ is unramified, the restriction of ρ_0 to the inertia subgroup has values into the elements of $GL_2(A)$ which are congruent to 1 modulo the maximal ideal of A. But the group of these elements is a p-group, and the wild inertia subgroup is a l-group. Therefore, ρ_0 is trivial on the wild inertia subgroup.

Let P_l be the wild inertia subgroup, and let I^t denote the group $I_{\mathbb{Q}_l}/P_l$. Then ρ_0 is determined by its values on I^t and on s, an element lifting the Frobenius element. Let $\phi = \rho_0(s)$; since ϕ lifts $\overline{\rho}(Frob_l)$, it has two distinct eigenvalues and is therefore diagonalizable. Moreover, we have for $t \in I^t$, $sts^{-1} = t^l$.

Let $t \in I^t$, and let $\tau = \rho_0(t)$. We will show that τ and ϕ have a common eigenvector. If it is not the case, let u be an eigenvector for τ for the eigenvalue λ. Then $\phi^{-1}(u)$ is an eigenvector of τ for the eigenvalue λ^l. Since $\phi^{-1}(u)$ is not colinear to u, τ is diagonalizable with eigenvalues λ and λ^l. The relation $\phi\tau\phi^{-1} = \tau^l$ shows then that $\lambda^{l^2} = \lambda$, and thus $\lambda^{l^2-1} = 1$. Since λ is congruent to 1 modulo the maximal ideal of A, and p is prime to $l+1$ (here we need $p \neq 2$), then by Hensel's lemma $\lambda^{l-1} = 1$. We deduce that $\lambda^l = \lambda$, and then τ is a scalar matrix.

We have shown that ϕ and τ have a common eigenvector. We can then suppose that $\phi = \begin{pmatrix} \alpha_l & 0 \\ 0 & \beta_l \end{pmatrix}$

and $\tau = \begin{pmatrix} \lambda & b \\ 0 & \mu \end{pmatrix}$. The relation $\phi\tau\phi^{-1} = \tau^l$, the fact that $\overline{\alpha_l}$ and $\overline{\beta_l}$ are distinct, and the congruence $l \equiv 1\ (p)$ allow us to conclude that $b = 0$. □

Remark 2.3. *In the case $p = 2$, the result is still valid (one shows that if $\lambda^l \neq \lambda$, then the trace of ϕ must be 0, which is impossible since $\overline{\alpha_l} + \overline{\beta_l} = \overline{\alpha_l} - \overline{\beta_l} \neq 0$).*
Another proof consists in wrinting $\phi = \begin{pmatrix} \alpha_l & 0 \\ 0 & \beta_l \end{pmatrix}$, $\tau = 1 + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with a, b, c, d in the maximal ideal of A. Then the relation $\phi\tau\phi^{-1} = \tau^l$ and Nakayama's lemma show that the non-diagonal terms b and c must be equal to 0.

The universal deformation of $\overline{\rho}|_{G_{\mathbb{Q}_l}}$ gives us two characters α_l and β_l lifting respectively $\overline{\alpha_l}$ and $\overline{\beta_l}$. The character $\alpha_l|_{I_{\mathbb{Q}_l}} : I_{\mathbb{Q}_l} \to R_l^\times$ gives by class field theory a morphism $\mathbb{Z}_l^\times \to R_l^\times$. Since the character is tamely ramified, it factors through $(\mathbb{Z}/l\mathbb{Z})^\times \to R_l^\times$. Let Δ_l be the p-Sylow subgroup of $(\mathbb{Z}/l\mathbb{Z})^\times$ (which is non trivial because of the congruence verified by l). We have a morphism $\Delta_l \to R_l^\times$. The ring R_l is then naturally an $\mathcal{O}[\Delta_l]$-algebra.

3 Global deformation rings

We have studied the deformations of $\overline{\rho}$ restricted to $G_{\mathbb{Q}_p}$, and to $G_{\mathbb{Q}_l}$ for a Taylo-Wiles prime l, and get rings $R_p^\square$ and R_l. We will also denote by $R_q^\square$ the ring of framed deformations of $\overline{\rho}|_{G_{\mathbb{Q}_q}}$ for a prime $q \in \Sigma$.
We will now study the global deformations of $\overline{\rho}$.

Definition 3.1. *Let $\mathbb{D} : \mathcal{A} \to Sets$ be the functor which assigns to $A \in \mathcal{A}$ the set of the deformations of $\overline{\rho}$ unramified outside Σ_p and with determinant ψ.*

An element of $\mathbb{D}(A)$ is a representation $\rho_0 : G_{\mathbb{Q}} \to GL_2(A)$ lifting $\overline{\rho}$, unramified outside $\Sigma_p = \Sigma \cup \{p\}$, and with det $\rho_0 = \psi$. Two such representations ρ_0 and ρ_0' are equivalent if there exists $h \in GL_2(A)$ congruent to 1 modulo the maximal ideal of A with $\rho_0 = h\rho_0' h^{-1}$.

Proposition 3.2. *The functor $\mathbb{D}$ is represented by a ring R.*

There is a universal representation $\rho^{univ} : G_{\mathbb{Q}} \to GL_2(R)$. For any set of primes S, we will denote by $\mathbb{Q}_S$ the maximal extension of $\mathbb{Q}$ unramified outside S, and $G_{\mathbb{Q},S} = Gal(\mathbb{Q}_S/\mathbb{Q})$. Since ρ^{univ} is unramified outside Σ_p, it factors through $G_{\mathbb{Q},\Sigma_p}$.
We will now compute the tangent space of R, that is to say the set $\mathbb{D}(\mathbb{F}[\epsilon])$. Let $V = \mathbb{F}^2$, so that we have $\overline{\rho} : G_{\mathbb{Q}} \to GL(V)$. An element in the tangent space is a morphism $\rho_1 : G_{\mathbb{Q}} \to GL_2(\mathbb{F}[\epsilon])$ such that

- ρ_1 is equal to $\overline{\rho}$ modulo ϵ.
- ρ_1 is unramified outside Σ_p, i.e. factors through $G_{\mathbb{Q},\Sigma_p}$.
- det $\rho_1 = \psi$.

For all $g \in G_{\mathbb{Q},\Sigma_p}$, write $\rho_1(g) = (1 + \epsilon f(g))\overline{\rho}(g)$, with $f(g) \in \mathrm{Ad}\,\overline{\rho} := Hom_{\mathbb{F}}(V, V)$. The fact that det $(1 + \epsilon f(g)) = 1$ implies that $f(g)$ belongs to $\mathrm{Ad}^0\,\overline{\rho}$, the subspace of $\mathrm{Ad}\,\overline{\rho}$ consisting of the elements of trace zero. The fact that ρ_1 is a morphism gives us the relations

$$f(g_1 g_2) = f(g_1) + \overline{\rho}(g_1) f(g_2) \overline{\rho}(g_1)^{-1}$$

for all $g_1, g_2 \in G_{\mathbb{Q},\Sigma_p}$. If we endow the space $\mathrm{Ad}\,\overline{\rho}$ with the action of $G_{\mathbb{Q},\Sigma_p}$ define by $g \cdot f = \overline{\rho}(g) f \overline{\rho}(g)^{-1}$ for $f \in \mathrm{Ad}\,\overline{\rho}$ and $g \in G_{\mathbb{Q},\Sigma_p}$ (this is the standard action on the space of morphism between representations), then we see that $\mathrm{Ad}^0\,\overline{\rho}$ is stable under that action, and that $f \in Z^1(G_{\mathbb{Q},\Sigma_p}, \mathrm{Ad}^0\,\overline{\rho})$.

Proposition 3.3. *Let h_1 be the dimension of the $\mathbb{F}$-vector space $H^1(G_{\mathbb{Q},\Sigma_p}, Ad^0\ \overline{\rho})$. Then R is generated over $\mathcal{O}$ by h_1 elements.*

Proof. We have seen that a element in the tangent space has the form $\rho_1 = (1+\epsilon f)\overline{\rho}$, with $f \in Z^1(G_{\mathbb{Q},\Sigma_p}, \mathrm{Ad}^0\ \overline{\rho})$. This representation is equivalent to $\rho_1' = (1-\epsilon h)\rho_1(1+\epsilon h)$, with $h \in \mathrm{Ad}\ \overline{\rho}$ (but up to the addition of a scalar matrix, we can suppose $h \in \mathrm{Ad}^0\ \overline{\rho}$). The cocycle f is equivalent to the cocycle f' defined by $f'(g) = f(g) + g\cdot h - h$. The tangent space $\mathbb{D}(\mathbb{F}[\epsilon])$ is thus isomorphic to $\mathrm{H}^1(G_{\mathbb{Q},\Sigma_p}, \mathrm{Ad}^0\ \overline{\rho})$. Since the number of generators is bounded by the dimension of the tangent space, the result follows. □

Remark 3.4. *The number of relations is bounded by the dimension of $H^2(G_{\mathbb{Q},\Sigma_p}, Ad^0\ \overline{\rho})$.*

We have defined the global deformation ring. We will relate this ring to the local deformation rings introduced in the first parts. First, we have to modify slighty the global deformation ring.

Definition 3.5. *Let $\mathbb{D}^\square : \mathcal{A} \to Sets$ be the functor which assigns to $A \in \mathcal{A}$ a tuple $(\rho_0, M_q, q \in \Sigma_p)$ where ρ_0 is a deformation of $\overline{\rho}$, unramified outside Σ_p with fixed determinant, and M_q is a frame for ρ_0 at q.*

An element of $\mathbb{D}^\square(A)$ is a representation $\rho_0 : G_{\mathbb{Q}} \to GL_2(A)$ lifting $\overline{\rho}$, unramified outside Σ_p, and with $\det \rho_0 = \psi$. Two such representations ρ_0 and ρ_0' are equivalent if there exists $h \in GL_2(A)$ congruent to 1 modulo the maximal ideal of A with $\rho_0 = h\rho_0' h^{-1}$, and if moreover the restrictions of ρ_0 and ρ_0' to $G_{\mathbb{Q}_q}$ are equal, for all $q \in \Sigma_p$.
The restriction of an element in $\mathbb{D}^\square(A)$ to $G_{\mathbb{Q}_q}$ gives a local framed deformation, for $q \in \Sigma_p$. We thus get a map $R_q^\square \to R^\square$, for $q \in \Sigma_p$, and thus $R^\square$ is a $R_{loc}^\square := R_p^\square \widehat{\otimes}_{q\in\Sigma} R_q^\square$-algebra.
Define

$$\mathrm{H}^1 = \mathrm{Ker}\left(\mathrm{H}^1(G_{\mathbb{Q},\Sigma_p}, \mathrm{Ad}^0\ \overline{\rho}) \to \oplus_{q\in\Sigma_p} \mathrm{H}^1(G_{\mathbb{Q}_q}, \mathrm{Ad}^0\ \overline{\rho})\right)$$

and let $h^1 = \dim_{\mathbb{F}} \mathrm{H}^1$.

Proposition 3.6. *The algebra $R^\square$ is generated over $R_{loc}^\square$ by $h^1 + |\Sigma_p| - 1$ elements.*

Let Q be a set of Taylor-Wiles primes. We note $\mathbb{D}_Q$ the functor of deformations of $\overline{\rho}$, unramified outside $Q \cup \Sigma_p$ with determinant ψ. This functor is represented by a ring R_Q, which is a R_l-algebra, for all $l \in Q$. The ring R_Q is thus an algebra over $\prod_{l\in Q} \mathcal{O}[\Delta_l] =: \mathcal{O}[\Delta_Q]$.
We also define $\mathbb{D}_Q^\square$ to be the functor of deformations of $\overline{\rho}$, unramified outside $Q \cup \Sigma_p$ with determinant ψ, together with frames at primes in Σ_p. It is represented by a ring $R_Q^\square$, which is an algebra over $R_{loc}^\square$. Define

$$\mathrm{H}_Q^1 = \mathrm{Ker}\left(\mathrm{H}^1(G_{\mathbb{Q},\Sigma_p\cup Q}, \mathrm{Ad}^0\ \overline{\rho}) \to \oplus_{q\in\Sigma_p} \mathrm{H}^1(G_{\mathbb{Q}_q}, \mathrm{Ad}^0\ \overline{\rho})\right)$$

and let $h_Q^1 = \dim_{\mathbb{F}} \mathrm{H}_Q^1$.

Proposition 3.7. *The algebra $R_Q^\square$ is generated over $R_{loc}^\square$ by $h_Q^1 + |\Sigma_p| - 1$ elements.*

Define

$$\mathrm{H}_\perp^1 = \mathrm{H}^1(G_{\mathbb{Q},\Sigma_p}, \mathrm{Ad}^0\ \overline{\rho}(1))$$

where $\overline{\rho}(1)$ is the Tate twist of $\overline{\rho}$, and

$$\mathrm{H}_{\perp,Q}^1 = \mathrm{Ker}\left(\mathrm{H}^1(G_{\mathbb{Q},\Sigma_p\cup Q}, \mathrm{Ad}^0\ \overline{\rho}(1)) \to \oplus_{q\in\Sigma_p} \mathrm{H}^1(G_{\mathbb{Q}_q}, \mathrm{Ad}^0\ \overline{\rho}(1))\right)$$

We will note $h_\perp^1 = \dim_{\mathbb{F}} \mathrm{H}_\perp^1$ and $h_{\perp,Q}^1 = \dim_{\mathbb{F}} \mathrm{H}_{\perp,Q}^1$. More generally, we will denote by $h^i(-)$ the $\mathbb{F}$-dimension of a cohomology group $H^i(-)$. The Poitou-Tate formula gives

$$h_Q^1 - h_{\perp,Q}^1 = h^0(G_{\mathbb{Q}}, \mathrm{Ad}^0\ \overline{\rho}) - h^0(G_{\mathbb{Q}}, \mathrm{Ad}^0\ \overline{\rho}(1)) + \sum_{l\in Q} h^2(G_{\mathbb{Q}_l}, \mathrm{Ad}^0\ \overline{\rho}) - h^0(G_\infty, \mathrm{Ad}^0\ \overline{\rho})$$

with $G_\infty = Gal(\mathbb{C}/\mathbb{R})$.

Proposition 3.8. *We have* $h^1_Q - h^1_{\perp,Q} = |Q| - 1$.

Proof. The space $H^0(G_\mathbb{Q}, \mathrm{Ad}^0\ \overline{\rho})$ consists of the elements of $\mathrm{Ad}^0\ \overline{\rho}$ fixed by $G_\mathbb{Q}$, i.e. the endomrorphisms of trace zero commuting with $\overline{\rho}$. Since $\overline{\rho}$ is absolutely irreducible, we have $h^0(G_\mathbb{Q}, \mathrm{Ad}^0\ \overline{\rho}) = 0$. Similarly, we have $h^0(G_\mathbb{Q}, \mathrm{Ad}^0\ \overline{\rho}(1)) = 0$. For $l \in Q$, we have by Galois duality $h^2(G_{\mathbb{Q}_l}, \mathrm{Ad}^0\ \overline{\rho}) = h^0(G_{\mathbb{Q}_l}, \mathrm{Ad}^0\ \overline{\rho}(1))$. Since $\overline{\rho}(1)$ restricted to $G_{\mathbb{Q}_l}$ is the sum of two distinct characters, we have $h^0(G_{\mathbb{Q}_l}, \mathrm{Ad}^0\ \overline{\rho}(1)) = 1$. Finally, since $\overline{\rho}$ is odd, we have that $\overline{\rho}(c)$ is conjugated to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, where c is the complex conjugation. By consequence, we have $h^0(G_\infty, \mathrm{Ad}^0\ \overline{\rho}) = 1$. □

It is possible to construct systems of Taylor-Wiles primes, which will be more and more precise.

Theorem 3.9. *For all* $n \geq 1$*, there exists a set of Taylor-Wiles primes* Q_n *such that*

- $|Q_n| = h^1_\perp$.
- $\forall l \in Q_n,\ l \equiv 1\ (p^n)$.
- $h^1_{\perp,Q_n} = 0$.

Consequently, we have $h^1_{Q_n} = h^1_{\perp,Q_n} + |Q_n| - 1 = h^1_\perp - 1$.

Corollary 3.10. *The algebra* $R^\square_{Q_n}$ *is generated over* $R^\square_{loc}$ *by* $h^1_\perp + |\Sigma_p| - 2$ *elements.*

The important thing is that the number of generators stays the same when n varies.

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